Prove the following inequality holds for the natural number $n \geq 2$:

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}}>\sqrt{n}$$

Proof.

We use mathematical induction to prove this inequality.

For the base case $n=2$ we need to show

$$1+\frac{1}{\sqrt{2}}>\sqrt{2}$$

Since $\sqrt{2}<2$ so this implies $\frac{1}{\sqrt{2}}>\frac{1}{2}$. Adding 1 to both sides gives

$$1+\frac{1}{\sqrt{2}}>1+\frac{1}{2} \qquad (*)$$

We use the arithmetic – geometric mean inequality which is given by

$$\frac{a+b}{2} \geq \sqrt{ab} \text{ provided } a \geq 0,\ b \geq 0$$

Applying this to the right hand side of the inequality (*) yields

$$1+\frac{1}{\sqrt{2}}>1+\frac{1}{2}=\frac{2+1}{2} \geq \sqrt{2\times 1}=\sqrt{2}$$

Hence we have our required inequality $1+\frac{1}{\sqrt{2}}>\sqrt{2}$.

Thus the base case $n=2$ is true.

Assume the inequality is true for $n=k$:

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{k}}>\sqrt{k} \qquad (\dagger)$$

Required to prove the result for $n=k+1$:

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{k+1}}>\sqrt{k+1}$$

Consider the left hand side of this inequality

$$\underbrace{1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{k}}}_{>\sqrt{k} \text{ by } (\dagger)}+\frac{1}{\sqrt{k+1}}>\sqrt{k}+\frac{1}{\sqrt{k+1}} \qquad (**)$$

Writing the right hand inequality with a common denominator gives

$$
\begin{aligned}
\sqrt{k}+\frac{1}{\sqrt{k+1}} &= \frac{\sqrt{k}\sqrt{k+1}}{\sqrt{k+1}}+\frac{1}{\sqrt{k+1}} \\
&= \frac{\sqrt{k}\sqrt{k+1}+1}{\sqrt{k+1}} \\
&\underbrace{>}_{\substack{\text{because} \\ \sqrt{k+1}>\sqrt{k}}} \frac{\sqrt{k}\sqrt{k}+1}{\sqrt{k+1}} = \frac{k+1}{\sqrt{k+1}} = \sqrt{k+1} \qquad \left[\begin{array}{l}\text{By the rules} \\ \text{of indices}\end{array}\right]
\end{aligned}
$$

Substituting this inequality $\sqrt{k}+\frac{1}{\sqrt{k+1}}>\sqrt{k+1}$ into (**) gives

$$
1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{k+1}}>\sqrt{k}+\frac{1}{\sqrt{k+1}}>\sqrt{k+1}
$$

Hence the inequality is true for $n=k+1$.

By mathematical induction we have our proven our given result.

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