Complete Solutions to Problems on Chapter 4 (Only available to tutors).

1. We apply the Gram Schmidt Process:

> Gram Schmidt Process (4.16).
>
> Let $\mathbf{p}_1 = \mathbf{v}_1$ and $\mathbf{p}_2 = \mathbf{v}_2 - \dfrac{\langle \mathbf{v}_2,\ \mathbf{p}_1 \rangle}{\|\mathbf{p}_1\|^2}\mathbf{p}_1$

Let $\mathbf{p}_1 = \mathbf{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\mathbf{v}_2 = \mathbf{v} = \begin{pmatrix} 0.999 \\ 0.001 \end{pmatrix}$. Then

$$\mathbf{p}_2 = \begin{pmatrix} 0.999 \\ 0.001 \end{pmatrix} - \frac{\begin{pmatrix} 0.999 \\ 0.001 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}}{\left\| \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\|^2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.999 \\ 0.001 \end{pmatrix} - \frac{0.999}{1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0.999 - 0.999 \\ 0.001 - 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0.001 \end{pmatrix} = \mathbf{p}_2$$

The first vector $\mathbf{p}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is already a unit vector. We need to convert $\mathbf{p}_2 = \begin{pmatrix} 0 \\ 0.001 \end{pmatrix}$ into a unit vector:

$$\hat{\mathbf{p}}_2 = \frac{1}{\sqrt{0^2 + 0.001^2}} \begin{pmatrix} 0 \\ 0.001 \end{pmatrix} = \frac{1}{0.001} \begin{pmatrix} 0 \\ 0.001 \end{pmatrix} = 1000 \begin{pmatrix} 0 \\ 0.001 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Our orthonormal vectors $\left\{ \mathbf{p}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \hat{\mathbf{p}}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ is our standard basis for $\mathbb{R}^2$.

We need to write the vector $\mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ in terms of the basis $\left\{ \mathbf{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \mathbf{v} = \begin{pmatrix} 0.999 \\ 0.001 \end{pmatrix} \right\}$:

$$\mathbf{x} = k\mathbf{u} + c\mathbf{v}$$

$$= k\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 0.999 \\ 0.001 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \Rightarrow \quad c = 1000,\ k = -998$$

Hence $\begin{pmatrix} 1 \\ 1 \end{pmatrix} = -998\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1000\begin{pmatrix} 0.999 \\ 0.001 \end{pmatrix}$.

Note that the given basis is not a very useful basis to work with.

2. Since **Q** is an orthogonal matrix so $\mathbf{Q}^T = \mathbf{Q}^{-1}$. Substituting $\mathbf{A} = \mathbf{QB}$ into the LHS of the above gives

$$\begin{aligned}\mathbf{A}^T\mathbf{A} &= (\mathbf{QB})^T\,\mathbf{QB}\\ &= \left(\mathbf{B}^T\mathbf{Q}^T\right)\mathbf{QB} \qquad \left[\text{Because } (\mathbf{XY})^T = \mathbf{Y}^T\mathbf{X}^T\right]\\ &= \mathbf{B}^T\left(\mathbf{Q}^T\mathbf{Q}\right)\mathbf{B} = \mathbf{B}^T(\mathbf{I})\mathbf{B} = \mathbf{B}^T\mathbf{B}\end{aligned}$$

3. (a) We have

$$\begin{aligned}\|\mathbf{A}\|_F^{\;2} = \sum_{j=1}^{2}\sum_{i=1}^{2}\left(a_{ij}\right)^2 &= (a_{11})^2 + (a_{21})^2 + (a_{12})^2 + (a_{22})^2\\ &= 1^2 + 3^2 + 2^2 + 4^2 = 30\end{aligned}$$

Taking the square root gives $\|\mathbf{A}\|_F = \sqrt{30}$.

(b) Similarly we have

$$\|\mathbf{A}\|_F^{\;2} = \sum_{j=1}^{5}\sum_{i=1}^{2}\left(a_{ij}\right)^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2 + 10^2 = 385$$

Taking the square root gives $\|\mathbf{A}\|_F = \sqrt{385} = 19.62\ (2\text{ dp})$.

4. We can check that **A** is an orthogonal matrix by using the following result:

Proposition (4.19). **Q** is an orthogonal matrix $\Leftrightarrow$ $\mathbf{Q}^T\mathbf{Q} = \mathbf{I}$

We carry out the matrix multiplication $\mathbf{A}^T\mathbf{A}$ and show that this is equal to the identity matrix **I**.

$$\mathbf{A}^T = \begin{pmatrix}\cos(\text{¯}) & -\sin(\text{¯})\\ \sin(\text{¯}) & \cos(\text{¯})\end{pmatrix}^T = \begin{pmatrix}\cos(\text{¯}) & \sin(\text{¯})\\ -\sin(\text{¯}) & \cos(\text{¯})\end{pmatrix}$$

Multiplying the 2 matrices **A** and $\mathbf{A}^T$ gives

$$\begin{aligned}\mathbf{A}^T\mathbf{A} &= \begin{pmatrix}\cos(\text{¯}) & \sin(\text{¯})\\ -\sin(\text{¯}) & \cos(\text{¯})\end{pmatrix}\begin{pmatrix}\cos(\text{¯}) & -\sin(\text{¯})\\ \sin(\text{¯}) & \cos(\text{¯})\end{pmatrix}\\ &= \begin{pmatrix}\cos^2(\text{¯}) + \sin^2(\text{¯}) & 0\\ 0 & \sin^2(\text{¯}) + \cos^2(\text{¯})\end{pmatrix}\\ &= \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \mathbf{I} \qquad \left[\text{Because } \cos^2(\text{¯}) + \sin^2(\text{¯}) = 1\right]\end{aligned}$$

Hence by Proposition (4.19) the above result $\mathbf{A}^T\mathbf{A} = \mathbf{I}$ $\Leftrightarrow$ $\mathbf{A}$ is an orthogonal matrix.

Also for an orthogonal matrix

$$\mathbf{A}^{-1} = \mathbf{A}^T = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$$

5. We need to see if the dot product of column vectors of **A** are equal to zero:

$$\begin{pmatrix} \cos(\theta) \\ 0 \\ -\sin(\theta) \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0, \quad \begin{pmatrix} \cos(\theta) \\ 0 \\ -\sin(\theta) \end{pmatrix} \cdot \begin{pmatrix} \sin(\theta) \\ 0 \\ \cos(\theta) \end{pmatrix} = \cos(\theta)\sin(\theta) - \sin(\theta)\cos(\theta) = 0, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \sin(\theta) \\ 0 \\ \cos(\theta) \end{pmatrix} = 0$$

Column vectors of matrix **A** are orthogonal to each other.

We also need to check that each column vector has a norm (length) of 1:

$$\left\| \begin{pmatrix} \cos(\theta) \\ 0 \\ -\sin(\theta) \end{pmatrix} \right\|^2 = \cos^2(\theta) + \left[-\sin(\theta)\right]^2 = 1, \quad \left\| \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\|^2 = 1, \quad \left\| \begin{pmatrix} \sin(\theta) \\ 0 \\ \cos(\theta) \end{pmatrix} \right\|^2 = \sin^2(\theta) + \cos^2(\theta) = 1$$

Taking the square root of each of these gives us a norm of 1 in each case.

Hence **A** is an orthogonal matrix and

$$\mathbf{A}^{-1} = \mathbf{A}^T = \begin{pmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{pmatrix}^T = \begin{pmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{pmatrix}$$

Evaluating **Ax**:

$$\mathbf{Ax} = \begin{pmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos(\theta) + \sin(\theta) \\ 1 \\ -\sin(\theta) + \cos(\theta) \end{pmatrix}$$

(a) Substituting $\theta = 90°$ into the above gives

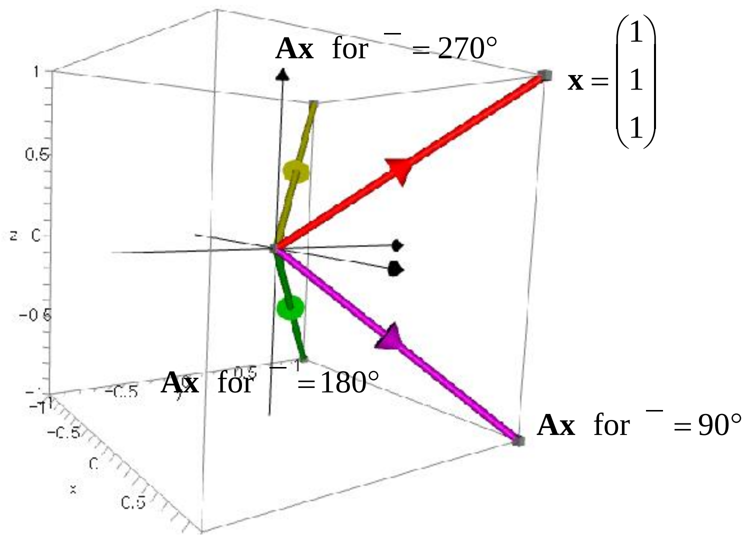
$$\mathbf{Ax} = \begin{pmatrix} \cos(90°) + \sin(90°) \\ 1 \\ -\sin(90°) + \cos(90°) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

(b) and (c). Similarly we have

$$\mathbf{Ax} = \begin{pmatrix} \cos(180°) + \sin(180°) \\ 1 \\ -\sin(180°) + \cos(180°) \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$$

$$\mathbf{Ax} = \begin{pmatrix} \cos(270°) + \sin(270°) \\ 1 \\ -\sin(270°) + \cos(270°) \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

The 3d picture of these vectors are



The matrix **A** rotates the vector **x** by θ° in an anti-clockwise direction.

6. For $\theta = 45°$ we have the matrix

$$\mathbf{A} = \begin{pmatrix} \cos(45°) & 0 & \sin(45°) \\ 0 & 1 & 0 \\ -\sin(45°) & 0 & \cos(45°) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

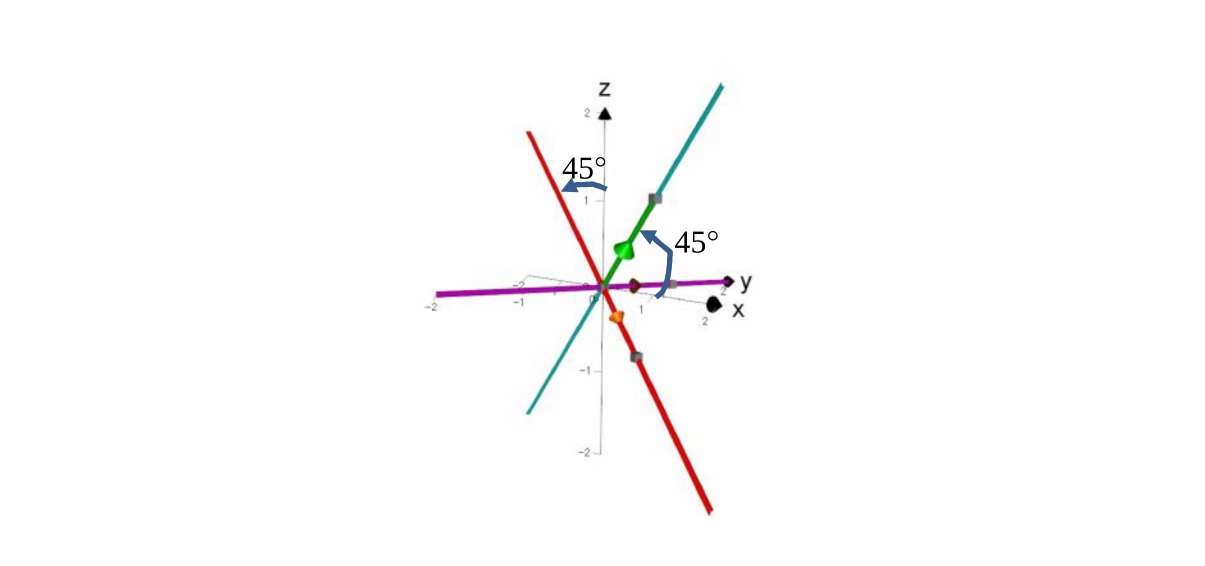
Applying this matrix **A** to each of the basis (axes) vectors $\{\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3\}$ we have

$$\mathbf{Ae}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\mathbf{Ae}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \sqrt{2} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\mathbf{Ae}_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

These new axes are sketched below:



The new axes are the thick lines with arrows. The $\mathbf{e}_1$ (*x*) and $\mathbf{e}_3$ (*z*) basis or axes vectors have been rotated by $45°$ in an anti-clockwise direction whilst the $\mathbf{e}_2$ (*y*) axis has remained fixed.

7. We can check that the functions 1 and *x* are already orthogonal because

$$\langle 1,\ x\rangle = \int_{-1}^{1} x\,\mathrm{d}x = \frac{1}{2}\left[x^2\right]_{-1}^{1} = 0$$

Need to normalize both these functions:

$$\|1\|^2 = \langle 1,\ 1\rangle = \int_{-1}^{1} 1\mathrm{d}x = \left[x\right]_{-1}^{1} = 2$$

Taking the square root gives $\|1\| = \sqrt{2}$. Similarly

$$\|x\|^2 = \langle x,\ x\rangle = \int_{-1}^{1} x^2\mathrm{d}x = \frac{1}{3}\left[x^3\right]_{-1}^{1} = \frac{2}{3} \quad \Rightarrow \quad \|x\| = \sqrt{\frac{2}{3}}$$

An orthonormal basis for P_1 is given by dividing the vectors by their lengths (norms); $\left\{\frac{1}{\sqrt{2}},\ \sqrt{\frac{3}{2}}x\right\}$.

8. We check the dot products of column vectors of the given matrix **A** are equal to zero (we don't need to worry about the half outside because if dot product is zero then multiplying by a half makes no difference):

$$\begin{pmatrix}1\\1\\1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\1\\-1\\-1\end{pmatrix}=0,\ \begin{pmatrix}1\\1\\1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}=0,\ \begin{pmatrix}1\\1\\1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\-1\\1\end{pmatrix}=0,\ \begin{pmatrix}1\\1\\-1\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}=0,\ \begin{pmatrix}1\\1\\-1\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\-1\\1\end{pmatrix}=0,\ \begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\-1\\1\end{pmatrix}=0$$

The length or norm of each column vector is 1 because we have a ½ on the outside. Hence the matrix **A** is orthogonal.

For an orthogonal matrix $\mathbf{A}^{-1} = \mathbf{A}^T$. Note that matrix **A** is symmetrical so $\mathbf{A}^T = \mathbf{A}$. The inverse of **A** is given by $\mathbf{A}^{-1} = \mathbf{A}^T = \mathbf{A}$.

9. The error is in the first line because the norm should be squared:

$$\left\|\mathbf{u}+\mathbf{v}-(\mathbf{u}-\mathbf{v})\right\|^2 = \langle 2\mathbf{v},\ 2\mathbf{v}\rangle$$

Actually our given equation should read $\left\|\mathbf{u}+\mathbf{v}-(\mathbf{u}-\mathbf{v})\right\| = \sqrt{\langle 2\mathbf{v},\ 2\mathbf{v}\rangle}$.

10. (i) Since the dimension of $\mathbb{R}^2$ is 2 so we need a vector **v** which is orthogonal to $\mathbf{u} = \begin{pmatrix} a \\ b \end{pmatrix}$ which means the dot product of **u** and **v** is equal to zero:

$$\mathbf{u}\cdot\mathbf{v} = \begin{pmatrix} a \\ b \end{pmatrix}\cdot\begin{pmatrix} x \\ y \end{pmatrix} = 0 \quad\Rightarrow\quad ax+by=0 \quad\Rightarrow\quad x = -\frac{b}{a}y \quad \text{provided } a \neq 0$$

Let $y = a$ then $x = -b$ and so $\mathbf{v} = \begin{pmatrix} -b \\ a \end{pmatrix}$. So far the vectors $\mathbf{u} = \begin{pmatrix} a \\ b \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} -b \\ a \end{pmatrix}$ are orthogonal but we need an orthonormal basis (axes). *How do we achieve this?* By normalizing each of these vectors:

$$\hat{\mathbf{u}} = \frac{1}{\sqrt{a^2+b^2}}\begin{pmatrix} a \\ b \end{pmatrix},\quad \hat{\mathbf{v}} = \frac{1}{\sqrt{a^2+b^2}}\begin{pmatrix} -b \\ a \end{pmatrix}$$

(ii) Substitute $a = b = 1$ into the vectors of part (i) gives

$$\hat{\mathbf{u}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix},\quad \hat{\mathbf{v}} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

11. We use the Gram Schmidt Process:

> Gram Schmidt Process (4.16).
>
> Let $\mathbf{p}_1 = \mathbf{v}_1$ then
>
> $$\mathbf{p}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2,\ \mathbf{p}_1\rangle}{\|\mathbf{p}_1\|^2}\mathbf{p}_1,\quad \mathbf{p}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3,\ \mathbf{p}_1\rangle}{\|\mathbf{p}_1\|^2}\mathbf{p}_1 - \frac{\langle \mathbf{v}_3,\ \mathbf{p}_2\rangle}{\|\mathbf{p}_2\|^2}\mathbf{p}_2$$

Applying this process to our given vectors:

$$\mathbf{p}_1 = \mathbf{u} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Let $\mathbf{v}_2 = \mathbf{v} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$ then

$$\mathbf{p}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2,\ \mathbf{p}_1 \rangle}{\|\mathbf{p}_1\|^2}\mathbf{p}_1 = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} - \frac{\begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}{\left\| \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\|^2} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} - \frac{2}{1}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2-2 \\ 0-0 \\ 3-0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$$

Similarly let $\mathbf{v}_3 = \mathbf{w} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$ then

$$\mathbf{p}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3,\ \mathbf{p}_1 \rangle}{\|\mathbf{p}_1\|^2}\mathbf{p}_1 - \frac{\langle \mathbf{v}_3,\ \mathbf{p}_2 \rangle}{\|\mathbf{p}_2\|^2}\mathbf{p}_2$$

$$= \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} - \frac{\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}{\left\| \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\|^2} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \frac{\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}}{\left\| \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} \right\|^2} \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} - \frac{4}{1}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \frac{18}{9}\begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 4-4-0 \\ 5-0-0 \\ 6-0-6 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix}$$

Our orthogonal vectors are $\mathbf{p}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{p}_2 = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$ and $\mathbf{p}_3 = \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix}$. Just need to normalize these in order to get an orthonormal basis. $\mathbf{p}_1$ is already of length 1.

$$\hat{\mathbf{p}}_2 = \frac{1}{3}\begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \text{ and } \hat{\mathbf{p}}_3 = \frac{1}{5}\begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Our orthogonal matrix $\mathbf{Q} = \begin{pmatrix} \mathbf{p}_1 & \hat{\mathbf{p}}_2 & \hat{\mathbf{p}}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. By

(4.26) $$\mathbf{R} = \mathbf{Q}^T\mathbf{A}$$

Applying this result to find the upper triangular matrix **R** we have

$$\mathbf{R} = \mathbf{Q}^T\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^T \begin{pmatrix} 1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 3 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 3 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 5 \end{pmatrix}$$

Hence the matrix **A** factorizes into

$$\mathbf{A}=\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}1&2&4\\0&3&6\\0&0&5\end{pmatrix}$$

12. (i) Let $\mathbf{u}=\begin{pmatrix}1\\2\\3\\4\end{pmatrix}$, $\mathbf{v}=\begin{pmatrix}1\\2\\0\\3\end{pmatrix}$ and $\mathbf{w}=\begin{pmatrix}2\\3\\5\\8\end{pmatrix}$. Applying the Gram Schmidt Process:

Gram Schmidt Process (4.16).Let $\mathbf{p}_1=\mathbf{v}_1$

$$\mathbf{p}_2=\mathbf{v}_2-\frac{\langle \mathbf{v}_2,\ \mathbf{p}_1\rangle}{\|\mathbf{p}_1\|^2}\mathbf{p}_1$$

$$\mathbf{p}_3=\mathbf{v}_3-\frac{\langle \mathbf{v}_3,\ \mathbf{p}_1\rangle}{\|\mathbf{p}_1\|^2}\mathbf{p}_1-\frac{\langle \mathbf{v}_3,\ \mathbf{p}_2\rangle}{\|\mathbf{p}_2\|^2}\mathbf{p}_2$$

We have $\mathbf{p}_1=\mathbf{u}=\begin{pmatrix}1\\2\\3\\4\end{pmatrix}$. Let $\mathbf{v}_2=\begin{pmatrix}1\\2\\0\\3\end{pmatrix}$ then

$$\mathbf{v}_2\cdot\mathbf{p}_1=\begin{pmatrix}1\\2\\0\\3\end{pmatrix}\cdot\begin{pmatrix}1\\2\\3\\4\end{pmatrix}=1+4+0+12=17 \text{ and } \|\mathbf{p}_1\|^2=\begin{pmatrix}1\\2\\3\\4\end{pmatrix}\cdot\begin{pmatrix}1\\2\\3\\4\end{pmatrix}=1^2+2^2+3^2+4^2=30$$

Substituting these values into $\mathbf{p}_2=\mathbf{v}_2-\frac{\langle \mathbf{v}_2,\ \mathbf{p}_1\rangle}{\|\mathbf{p}_1\|^2}\mathbf{p}_1$ gives

$$\mathbf{p}_2=\begin{pmatrix}1\\2\\0\\3\end{pmatrix}-\frac{17}{30}\begin{pmatrix}1\\2\\3\\4\end{pmatrix}=\begin{pmatrix}1-17/30\\2-34/30\\0-51/30\\3-68/30\end{pmatrix}=\begin{pmatrix}13/30\\26/30\\-51/30\\22/30\end{pmatrix}=\frac{1}{30}\begin{pmatrix}13\\26\\-51\\22\end{pmatrix}$$

For ease of arithmetic we ignore our fraction 1/30 and write

$$\mathbf{p}_2{*}=\begin{pmatrix}13\\26\\-51\\22\end{pmatrix}$$

Now we use $\mathbf{p}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3,\ \mathbf{p}_1 \rangle}{\|\mathbf{p}_1\|^2}\mathbf{p}_1 - \frac{\langle \mathbf{v}_3,\ \mathbf{p}_2 * \rangle}{\|\mathbf{p}_2 *\|^2}\mathbf{p}_2 *$ with $\mathbf{v}_3 = \begin{pmatrix} 2 \\ 3 \\ 5 \\ 8 \end{pmatrix}$. We have

$$\mathbf{v}_3 \cdot \mathbf{p}_1 = \begin{pmatrix} 2 \\ 3 \\ 5 \\ 8 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = 2 + 6 + 15 + 32 = 55$$

$$\mathbf{v}_3 \cdot \mathbf{p}_2 * = \begin{pmatrix} 13 \\ 26 \\ -51 \\ 22 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ 5 \\ 8 \end{pmatrix} = 26 + 78 - 255 + 176 = 25$$

$$\|\mathbf{p}_2 *\|^2 = \begin{pmatrix} 13 \\ 26 \\ -51 \\ 22 \end{pmatrix} \cdot \begin{pmatrix} 13 \\ 26 \\ -51 \\ 22 \end{pmatrix} = 13^2 + 26^2 + (-51)^2 + 22^2 = 3930$$

Putting each of these into $\mathbf{p}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3,\ \mathbf{p}_1 \rangle}{\|\mathbf{p}_1\|^2}\mathbf{p}_1 - \frac{\langle \mathbf{v}_3,\ \mathbf{p}_2 * \rangle}{\|\mathbf{p}_2 *\|^2}\mathbf{p}_2 *$ gives

$$\mathbf{p}_3 = \begin{pmatrix} 2 \\ 3 \\ 5 \\ 8 \end{pmatrix} - \frac{55}{30}\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} - \frac{25}{3930}\begin{pmatrix} 13 \\ 26 \\ -51 \\ 22 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 5 \\ 8 \end{pmatrix} - \frac{11}{6}\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} - \frac{5}{786}\begin{pmatrix} 13 \\ 26 \\ -51 \\ 22 \end{pmatrix} \quad [\text{Simplifying the fractions}]$$

Evaluating this gives

$$\mathbf{p}_3 = \begin{pmatrix} 2 \\ 3 \\ 5 \\ 8 \end{pmatrix} - \frac{11}{6}\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} - \frac{5}{786}\begin{pmatrix} 13 \\ 26 \\ -51 \\ 22 \end{pmatrix} = \begin{pmatrix} 2 - 11/6 - 65/786 \\ 3 - 22/6 - 130/786 \\ 5 - 33/6 + 255/786 \\ 8 - 44/6 - 110/786 \end{pmatrix} = \begin{pmatrix} 11/131 \\ -109/131 \\ -23/131 \\ 69/131 \end{pmatrix} = \frac{1}{131}\begin{pmatrix} 11 \\ -109 \\ -23 \\ 69 \end{pmatrix}$$

Ignore the fraction so that arithmetic is easier

$$\mathbf{p}_3 * = \begin{pmatrix} 11 \\ -109 \\ -23 \\ 69 \end{pmatrix}$$

You may like to check that $\mathbf{p}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$, $\mathbf{p}_2 * = \begin{pmatrix} 13 \\ 26 \\ -51 \\ 22 \end{pmatrix}$ and $\mathbf{p}_3 * = \begin{pmatrix} 11 \\ -109 \\ -23 \\ 69 \end{pmatrix}$ are orthogonal

by showing the dot product of each of these is zero.

We need to normalize each of these vectors to give us an orthonormal basis. The first two vectors have already been normalized:

$$\hat{\mathbf{p}}_1 = \frac{1}{\sqrt{30}}\begin{pmatrix}1\\2\\3\\4\end{pmatrix}, \quad \mathbf{p}_2{*} = \frac{1}{\sqrt{3930}}\begin{pmatrix}13\\26\\-51\\22\end{pmatrix}$$

Finding the norm of the last vector:

$$\|\mathbf{p}_3{*}\| = \sqrt{11^2 + (-109)^2 + (-23)^2 + 69^2} = \sqrt{17292}$$

We have

$$\widehat{\mathbf{p}_3}{*} = \frac{1}{\sqrt{17292}}\begin{pmatrix}11\\-109\\-23\\69\end{pmatrix}$$

Our orthonormal basis for the given subspace is

$$\left\{\frac{1}{\sqrt{30}}\begin{pmatrix}1\\2\\3\\4\end{pmatrix}, \frac{1}{\sqrt{3930}}\begin{pmatrix}13\\26\\-51\\22\end{pmatrix}, \frac{1}{\sqrt{17292}}\begin{pmatrix}11\\-109\\-23\\69\end{pmatrix}\right\}$$

(ii) We use the results of part (i) to write the given matrix into **QR** factorization:

$$\mathbf{Q} = \begin{pmatrix}\hat{\mathbf{p}}_1 & \hat{\mathbf{p}}_2{*} & \widehat{\mathbf{p}_3}{*}\end{pmatrix} = \begin{pmatrix}1/\sqrt{30} & 13/\sqrt{3930} & 11/\sqrt{17292}\\ 2/\sqrt{30} & 26/\sqrt{3930} & -109/\sqrt{17292}\\ 3/\sqrt{30} & -51/\sqrt{3930} & -23/\sqrt{17292}\\ 4/\sqrt{30} & 22/\sqrt{3930} & 69/\sqrt{17292}\end{pmatrix}$$

From chapter 4 we have

(4.26) $$\mathbf{R} = \mathbf{Q}^T\mathbf{A}$$

Applying this to the above matrix **Q** and given matrix **A**:

$$\mathbf{R} = \mathbf{Q}^T\mathbf{A}$$

$$= \begin{pmatrix} 1/\sqrt{30} & 13/\sqrt{3930} & 11/\sqrt{17292} \\ 2/\sqrt{30} & 26/\sqrt{3930} & -109/\sqrt{17292} \\ 3/\sqrt{30} & -51/\sqrt{3930} & -23/\sqrt{17292} \\ 4/\sqrt{30} & 22/\sqrt{3930} & 69/\sqrt{17292} \end{pmatrix}^T \begin{pmatrix} 1 & 1 & 2 \\ 2 & 2 & 3 \\ 3 & 0 & 5 \\ 4 & 3 & 8 \end{pmatrix}$$

$$= \begin{pmatrix} 1/\sqrt{30} & 2/\sqrt{30} & 3/\sqrt{30} & 4/\sqrt{30} \\ 13/\sqrt{3930} & 26/\sqrt{3930} & -51/\sqrt{3930} & 22/\sqrt{3930} \\ 11/\sqrt{17292} & -109/\sqrt{17292} & -23/\sqrt{17292} & 69/\sqrt{17292} \end{pmatrix} \begin{pmatrix} 1 & 1 & 2 \\ 2 & 2 & 3 \\ 3 & 0 & 5 \\ 4 & 3 & 8 \end{pmatrix}$$

$$= \begin{pmatrix} 30/\sqrt{30} & 17/\sqrt{30} & 55/\sqrt{30} \\ 0 & 131/\sqrt{3930} & 25/\sqrt{3930} \\ 0 & 0 & 132/\sqrt{17292} \end{pmatrix}$$

13. We use the Gram Schmidt Process on $\mathbf{u} = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 1/3 \\ 2/3 \\ 1 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 1/4 \\ 1/4 \\ 1/2 \end{pmatrix}$. Since these are mostly fractional entries so to make the calculations easier it is better to multiple by an appropriate factor to give simple integer entries:

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \text{ and } \mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

Gram Schmidt Process (4.16).

Let $\mathbf{p}_1 = \mathbf{v}_1$ then

$$\mathbf{p}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{p}_1 \rangle}{\|\mathbf{p}_1\|^2}\mathbf{p}_1, \quad \mathbf{p}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3, \mathbf{p}_1 \rangle}{\|\mathbf{p}_1\|^2}\mathbf{p}_1 - \frac{\langle \mathbf{v}_3, \mathbf{p}_2 \rangle}{\|\mathbf{p}_2\|^2}\mathbf{p}_2$$

Let $\mathbf{p}_1 = \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ then

$$\mathbf{p}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2,\ \mathbf{p}_1 \rangle}{\|\mathbf{p}_1\|^2}\mathbf{p}_1 = \begin{pmatrix}1\\2\\3\end{pmatrix} - \frac{\begin{pmatrix}1\\2\\3\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}}{\left\|\begin{pmatrix}1\\1\\1\end{pmatrix}\right\|^2}\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}1\\2\\3\end{pmatrix} - \frac{6}{3}\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}1-2\\2-2\\3-2\end{pmatrix} = \begin{pmatrix}-1\\0\\1\end{pmatrix}$$

Check that $\mathbf{p}_1$ and $\mathbf{p}_2$ are orthogonal (perpendicular).

$$\mathbf{p}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3,\ \mathbf{p}_1 \rangle}{\|\mathbf{p}_1\|^2}\mathbf{p}_1 - \frac{\langle \mathbf{v}_3,\ \mathbf{p}_2 \rangle}{\|\mathbf{p}_2\|^2}\mathbf{p}_2$$

$$= \begin{pmatrix}1\\1\\2\end{pmatrix} - \frac{\begin{pmatrix}1\\1\\2\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}}{\left\|\begin{pmatrix}1\\1\\1\end{pmatrix}\right\|^2}\begin{pmatrix}1\\1\\1\end{pmatrix} - \frac{\begin{pmatrix}1\\1\\2\end{pmatrix}\cdot\begin{pmatrix}-1\\0\\1\end{pmatrix}}{\left\|\begin{pmatrix}-1\\0\\1\end{pmatrix}\right\|^2}\begin{pmatrix}-1\\0\\1\end{pmatrix} = \begin{pmatrix}1\\1\\2\end{pmatrix} - \frac{4}{3}\begin{pmatrix}1\\1\\1\end{pmatrix} - \frac{1}{2}\begin{pmatrix}-1\\0\\1\end{pmatrix} = \begin{pmatrix}1-4/3+1/2\\1-4/3-0\\2-4/3-1/2\end{pmatrix} = \begin{pmatrix}1/6\\-1/3\\1/6\end{pmatrix}$$

Multiply the last vector by 6 so that we have simple integer entries $\mathbf{p}_3{*} = \begin{pmatrix}1\\-2\\1\end{pmatrix}$.

Our three vectors $\mathbf{p}_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}$, $\mathbf{p}_2 = \begin{pmatrix}-1\\0\\1\end{pmatrix}$ and $\mathbf{p}_3{*} = \begin{pmatrix}1\\-2\\1\end{pmatrix}$ are orthogonal (perpendicular) but we need to normalize these so that we get an orthonormal basis:

$$\hat{\mathbf{p}}_1 = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix},\ \hat{\mathbf{p}}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\0\\1\end{pmatrix},\ \hat{\mathbf{p}}_3{*} = \frac{1}{\sqrt{6}}\begin{pmatrix}1\\-2\\1\end{pmatrix}$$

Our orthonormal basis is $\left\{\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\0\\1\end{pmatrix}, \frac{1}{\sqrt{6}}\begin{pmatrix}1\\-2\\1\end{pmatrix}\right\}$.

14. We need to produce a counter example;

Let $\mathbf{u} = \begin{pmatrix}1 & 2 & 3 & 1\end{pmatrix}^T$ then using the given function

$$\left\langle \begin{pmatrix}x_1 & y_1 & z_1 & t_1\end{pmatrix}^T,\ \begin{pmatrix}x_2 & y_2 & z_2 & t_2\end{pmatrix}^T \right\rangle = -x_1x_2 - y_1y_2 - z_1z_2 + t_1t_2$$

We have

$$\left\langle \begin{pmatrix}1 & 2 & 3 & 1\end{pmatrix}^T,\ \begin{pmatrix}1 & 2 & 3 & 1\end{pmatrix}^T \right\rangle = -(1\times1) - (2\times2) - (3\times3) + (1\times1) = -13 < 0$$

Also for $\mathbf{v} = \begin{pmatrix}1 & 2 & 3 & \sqrt{14}\end{pmatrix}^T$ we have

$$\left\langle \begin{pmatrix}1 & 2 & 3 & \sqrt{14}\end{pmatrix}^T, \begin{pmatrix}1 & 2 & 3 & \sqrt{14}\end{pmatrix}^T \right\rangle = -(1\times 1)-(2\times 2)-(3\times 3)+\left(\sqrt{14}\times\sqrt{14}\right)=0$$

Hence we have $\langle \mathbf{v},\ \mathbf{v}\rangle = 0$ but $\mathbf{v} \neq \mathbf{O}$.

Using the given formula

$$d(\mathbf{u},\ \mathbf{v}) = \left\| \begin{pmatrix}u_1-v_1 & u_2-v_2 & u_3-v_3 & u_4-v_4\end{pmatrix}^T \right\|$$

We have

$$d(\mathbf{u},\ \mathbf{v}) = \left\| \begin{pmatrix}1-5 & 2-6 & 3-7 & 4-8\end{pmatrix}^T \right\| = \left\| \begin{pmatrix}4 & 4 & 4 & 4\end{pmatrix}^T \right\|$$

Using $\|\mathbf{u}\|^2 = |\langle \mathbf{u},\ \mathbf{u}\rangle|$ and

$$\left\langle \begin{pmatrix}x_1 & y_1 & z_1 & t_1\end{pmatrix}^T, \begin{pmatrix}x_2 & y_2 & z_2 & t_2\end{pmatrix}^T \right\rangle = -x_1x_2 - y_1y_2 - z_1z_2 + t_1t_2$$

We have

$$\left[d(\mathbf{u},\ \mathbf{v})\right]^2 = \left\| \begin{pmatrix}4 & 4 & 4 & 4\end{pmatrix}^T \right\|^2$$

$$= \left|-4^2-4^2-4^2+4^2\right| = |-32| = 32$$

Hence $d(\mathbf{u},\ \mathbf{v}) = \sqrt{32}$.

15. We use the Gram Schmidt Process:

Gram Schmidt Process (4.16).

Let $\mathbf{p}_1 = \mathbf{v}_1$ then

$$\mathbf{p}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2,\ \mathbf{p}_1\rangle}{\|\mathbf{p}_1\|^2}\mathbf{p}_1,\quad \mathbf{p}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3,\ \mathbf{p}_1\rangle}{\|\mathbf{p}_1\|^2}\mathbf{p}_1 - \frac{\langle \mathbf{v}_3,\ \mathbf{p}_2\rangle}{\|\mathbf{p}_2\|^2}\mathbf{p}_2$$

Using this we have $\mathbf{p}_1 = 1$. We find a polynomial which is orthogonal to this $\mathbf{p}_1 = 1$:

$$\mathbf{p}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2,\ \mathbf{p}_1\rangle}{\|\mathbf{p}_1\|^2}\mathbf{p}_1$$

Let $\mathbf{v}_2 = x$. Then

$$\langle \mathbf{v}_2,\ \mathbf{p}_1\rangle = \int_0^1 x(1)\,dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}$$

Substituting $\mathbf{v}_2 = x$, $\langle \mathbf{v}_2,\ \mathbf{p}_1\rangle = \frac{1}{2}$, $\mathbf{p}_1 = 1$ and $\|\mathbf{p}_1\|^2 = 1$ into $\mathbf{p}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2,\ \mathbf{p}_1\rangle}{\|\mathbf{p}_1\|^2}\mathbf{p}_1$:

$$\mathbf{p}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2,\ \mathbf{p}_1\rangle}{\|\mathbf{p}_1\|^2}\mathbf{p}_1 = x - \frac{1}{2}$$

To find the last vector we use $\mathbf{p}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3, \mathbf{p}_1 \rangle}{\|\mathbf{p}_1\|^2}\mathbf{p}_1 - \frac{\langle \mathbf{v}_3, \mathbf{p}_2 \rangle}{\|\mathbf{p}_2\|^2}\mathbf{p}_2$. Let $\mathbf{v}_3 = x^2$ then

$$\langle \mathbf{v}_3, \mathbf{p}_1 \rangle = \int_0^1 x^2(1)\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

$$\langle \mathbf{v}_3, \mathbf{p}_2 \rangle = \int_0^1 x^2\left(x - \frac{1}{2}\right)dx = \int_0^1 \left(x^3 - \frac{x^2}{2}\right)dx = \left[\frac{x^4}{4} - \frac{x^3}{6}\right]_0^1 = \frac{1}{4} - \frac{1}{6} = \frac{1}{12}$$

$$\|\mathbf{p}_2\|^2 = \int_0^1 \left(x - \frac{1}{2}\right)\left(x - \frac{1}{2}\right)dx = \int_0^1 \left(x^2 - x + \frac{1}{4}\right)dx$$

$$= \left[\frac{x^3}{3} - \frac{x^2}{2} + \frac{x}{4}\right]_0^1 = \frac{1}{3} - \frac{1}{2} + \frac{1}{4} = \frac{1}{12}$$

Substituting $\mathbf{v}_3 = x^2$, $\langle \mathbf{v}_3, \mathbf{p}_1 \rangle = \frac{1}{3}$, $\langle \mathbf{v}_3, \mathbf{p}_2 \rangle = \frac{1}{12}$, $\|\mathbf{p}_2\|^2 = \frac{1}{12}$ and $\|\mathbf{p}_1\|^2 = 1$ into

$\mathbf{p}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3, \mathbf{p}_1 \rangle}{\|\mathbf{p}_1\|^2}\mathbf{p}_1 - \frac{\langle \mathbf{v}_3, \mathbf{p}_2 \rangle}{\|\mathbf{p}_2\|^2}\mathbf{p}_2$:

$$\mathbf{p}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3, \mathbf{p}_1 \rangle}{\|\mathbf{p}_1\|^2}\mathbf{p}_1 - \frac{\langle \mathbf{v}_3, \mathbf{p}_2 \rangle}{\|\mathbf{p}_2\|^2}\mathbf{p}_2$$

$$= x^2 - \frac{1/3}{1}(1) - \frac{1/12}{1/12}\left(x - \frac{1}{2}\right) = x^2 - \frac{1}{3} - \left(x - \frac{1}{2}\right) = x^2 - x + \frac{1}{6}$$

We have our orthogonal basis $\mathbf{p}_1 = 1$, $\mathbf{p}_2 = x - \frac{1}{2}$ and $\mathbf{p}_3 = x^2 - x + \frac{1}{6}$. We just need to normalize these vectors.

Note $\mathbf{p}_1 = 1$ is already normalized. From above we already have $\|\mathbf{p}_2\|^2 = \frac{1}{12}$ so taking the square root gives $\|\mathbf{p}_2\| = \frac{1}{\sqrt{12}}$ which means our normalized vector is

$$\hat{\mathbf{p}}_2 = \sqrt{12}\left(x - \frac{1}{2}\right)$$

By using the given definition of the inner product:

$$\|\mathbf{p}_3\|^2 = \int_0^1 \left(x^2 - x + \frac{1}{6}\right)^2 dx = \frac{1}{180}$$

Hence $\hat{\mathbf{p}}_3 = \sqrt{180}\left(x^2 - x + \frac{1}{6}\right)$. Our orthonormal basis is

$$\left\{1,\ \sqrt{12}\left(x - \frac{1}{2}\right),\ \sqrt{180}\left(x^2 - x + \frac{1}{6}\right)\right\}$$

16. Let $p(x)=ax^3+bx^2+cx+d$ because we are looking for a cubic polynomial. This cubic polynomial must satisfy

$$\langle p,1\rangle=\langle p,\ x\rangle=\left\langle p,\ x^2-\frac{1}{3}\right\rangle=0$$

First we examine $\langle p,1\rangle=0$:

$$\langle p,1\rangle=\int_{-1}^{1}\left(ax^3+bx^2+cx+d\right)dx=\left[\frac{ax^4}{4}+\frac{bx^3}{3}+\frac{cx^2}{2}+dx\right]_{-1}^{1}$$
$$=\left[\frac{a}{4}+\frac{b}{3}+\frac{c}{2}+d\right]-\left[\frac{a}{4}-\frac{b}{3}+\frac{c}{2}-d\right]$$
$$=\frac{2b}{3}+2d=0$$

From the last line we have

$$\frac{2b}{3}+2d=0\ \Rightarrow\ \frac{2b}{3}=-2d\ \Rightarrow\ b=-3d \qquad (*)$$

Now we look at $\langle p,x\rangle=0$:

$$\langle p,x\rangle=\int_{-1}^{1}\left(ax^3+bx^2+cx+d\right)xdx=\int_{-1}^{1}\left(ax^4+bx^3+cx^2+dx\right)dx$$
$$=\left[\frac{ax^5}{5}+\frac{bx^4}{4}+\frac{cx^3}{3}+\frac{dx^2}{2}\right]_{-1}^{1}$$
$$=\left[\frac{a}{5}+\frac{b}{4}+\frac{c}{3}+\frac{d}{2}\right]-\left[-\frac{a}{5}+\frac{b}{4}-\frac{c}{3}+\frac{d}{2}\right]$$
$$=\frac{2a}{5}+\frac{2c}{3}=0$$

Similarly we have

$$\frac{2a}{5}+\frac{2c}{3}=0\ \Rightarrow\ \frac{2a}{5}=-\frac{2c}{3}\ \Rightarrow\ a=-\frac{5}{3}c \qquad (**)$$

Lastly we examine $\left\langle p,\ x^2-\frac{1}{3}\right\rangle=0$:

$$\left\langle p, x^2 - \frac{1}{3} \right\rangle = \int_{-1}^{1} \left(ax^3 + bx^2 + cx + d \right)\left(x^2 - \frac{1}{3} \right) dx$$

$$= \int_{-1}^{1} \left(ax^5 + bx^4 + cx^3 + dx^2 \right) dx - \frac{1}{3}\int_{-1}^{1} \left(ax^3 + bx^2 + cx + d \right) dx$$

$$= \left[\frac{ax^6}{6} + \frac{bx^5}{5} + \frac{cx^4}{4} + \frac{dx^3}{3} \right]_{-1}^{1} - \frac{1}{3}\left[\frac{2b}{3} + 2d \right] \qquad [\text{By above}]$$

$$= \left[\frac{a}{6} + \frac{b}{5} + \frac{c}{4} + \frac{d}{3} \right] - \left[\frac{a}{6} - \frac{b}{5} + \frac{c}{4} - \frac{d}{3} \right] - \frac{2}{3}\left[\frac{b}{3} + d \right]$$

$$= \left[\frac{2b}{5} + \frac{2d}{3} \right] - \frac{2}{3}\left[\frac{b}{3} + d \right] = \frac{2b}{5} - \frac{2b}{9} = 0 \quad \Rightarrow \quad b = 0$$

Substituting $b = 0$ into (*) gives $d = 0$. By (**) we have $a = -\frac{5}{3}c$. As long as this equation is satisfied we have ***p*** orthogonal to all our given vectors. Let $c = -\frac{3}{5}$ then $a = 1$.

Substituting $a = 1$, $b = 0$, $c = -\frac{3}{5}$ and $d = 0$ into $p(x) = ax^3 + bx^2 + cx + d$ gives

$$p(x) = x^3 - \frac{3}{5}x$$

17. (i) *How do we show that vectors **u**, **v** and **w** are orthogonal?*

Show that the dot product between each of the vectors is zero:

$$\mathbf{u}\cdot\mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix} = 2 + 2 - 4 = 0$$

$$\mathbf{u}\cdot\mathbf{w} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} = 3 - 4 + 1 = 0$$

$$\mathbf{v}\cdot\mathbf{w} = \begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} = 6 - 2 - 4 = 0$$

Hence the vectors in the set $\{\mathbf{u},\ \mathbf{v},\ \mathbf{w}\}$ are orthogonal. *How do show that this set is a basis for $\mathbb{R}^3$?*

Using the following result of chapter 4:

> Corollary (4.15). In an *n* dimensional inner product space *V*, any set of *n* orthogonal non – zero vectors forms a basis (or an axes) for *V*.

We conclude that $\{\mathbf{u},\ \mathbf{v},\ \mathbf{w}\}$ is an orthogonal basis for $\mathbb{R}^3$.

(ii) We need to find the values of the scalars $k_1,\ k_2,\ k_3$ such that

$$k_1\begin{pmatrix}1\\2\\1\end{pmatrix}+k_2\begin{pmatrix}2\\1\\-4\end{pmatrix}+k_3\begin{pmatrix}3\\-2\\1\end{pmatrix}=\begin{pmatrix}9\\-3\\9\end{pmatrix}$$

Writing the given vectors and the right hand side in an augmented matrix:

$$\begin{matrix}\mathrm{R}_1\\\mathrm{R}_2\\\mathrm{R}_3\end{matrix}\left(\begin{array}{ccc|c}1&2&3&9\\2&1&-2&-3\\1&-4&1&9\end{array}\right)$$

Carrying out the following row operations:

$$\begin{matrix}\mathrm{R}_1\\\mathrm{R}_2{*}=\mathrm{R}_2-2\mathrm{R}_1\\\mathrm{R}_3{*}=\mathrm{R}_3-\mathrm{R}_1\end{matrix}\left(\begin{array}{ccc|c}1&2&3&9\\0&-3&-8&-21\\0&-6&-2&0\end{array}\right)$$

Executing the following:

$$\begin{matrix}\\\mathrm{R}_1\\\mathrm{R}_2{*}\\\mathrm{R}_3{**}=\mathrm{R}_3{*}-2\mathrm{R}_2{*}\end{matrix}\begin{array}{c}\begin{matrix}k_1&k_2&k_3\end{matrix}\\\left(\begin{array}{ccc|c}1&2&3&9\\0&-3&-8&-21\\0&0&14&42\end{array}\right)\end{array}$$

From the bottom row we have

$$14k_3=42\ \Rightarrow\ k_3=3$$

Substituting $k_3=3$ into the middle row yields

$$-3k_2-8k_3=-21\ \Rightarrow\ -3k_2-8(3)=-21\ \Rightarrow\ k_2=-1$$

Substituting $k_2=-1$ and $k_3=3$ into the top row gives

$$k_1+2k_2+3k_3=9\ \Rightarrow\ k_1+2(-1)+3(3)=9\ \Rightarrow\ k_1=2$$

Hence $\mathbf{x}=2\mathbf{u}-\mathbf{v}+3\mathbf{w}=\begin{pmatrix}9&-3&9\end{pmatrix}^T$.

(iii) Using the given formula we have

$$c_1=\frac{\mathbf{x}\cdot\mathbf{u}}{\|\mathbf{u}\|^2}=\frac{\begin{pmatrix}9\\-3\\9\end{pmatrix}\cdot\begin{pmatrix}1\\2\\1\end{pmatrix}}{\left\|\begin{pmatrix}1\\2\\1\end{pmatrix}\right\|^2}=\frac{9-6+9}{1^2+2^2+1^2}=\frac{12}{6}=2$$

Similarly we have

$$c_2=\frac{\mathbf{x}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}=-1\ \text{ and }\ c_3=\frac{\mathbf{x}\cdot\mathbf{w}}{\|\mathbf{w}\|^2}=3$$

These are values of the scalars we found in part (ii). Hence

$$\mathbf{x} = 2\mathbf{u} - \mathbf{v} + 3\mathbf{w} = \begin{pmatrix} 9 & -3 & 9 \end{pmatrix}^T$$

18. The error is in the first line because the last term on the right should have a k^2 term:

$$\langle \mathbf{u} - k\mathbf{v},\ \mathbf{u} - k\mathbf{v} \rangle = \|\mathbf{u}\|^2 - 2k\langle \mathbf{u},\ \mathbf{v} \rangle + k^2\|\mathbf{v}\|^2$$

19. Since **Q** is an orthogonal matrix so the column vectors of **Q** are orthogonal:

$$\begin{pmatrix} a \\ c \end{pmatrix} \cdot \begin{pmatrix} b \\ d \end{pmatrix} = ab + cd = 0$$

Also the column vectors of **Q** are normalized vectors which means

$$a^2 + c^2 = 1 \quad \text{and} \quad b^2 + d^2 = 1$$

Using these results to evaluate $(a+b)^2 + (c+d)^2$ gives

$$(a+b)^2 + (c+d)^2 = (a^2 + 2ab + b^2) + (c^2 + 2cd + d^2)$$
$$= \underbrace{a^2 + c^2}_{=1} + \underbrace{b^2 + d^2}_{=1} + 2\underbrace{(ab + cd)}_{=0} = 1 + 1 = 2$$

20. There is no error because $\langle \mathbf{u},\ 2\mathbf{v} \rangle = 2\langle \mathbf{u},\ \mathbf{v} \rangle = \langle 2\mathbf{v},\ \mathbf{u} \rangle = 2\langle \mathbf{u},\ \mathbf{v} \rangle$.

21. We are given that **A** is an orthogonal matrix. By

> Proposition (4.19). **Q** is an orthogonal matrix $\Leftrightarrow$ $\mathbf{Q}^T\mathbf{Q} = \mathbf{I}$

We have $\mathbf{A}^T\mathbf{A} = \mathbf{I}$.

Required to prove that matrix $\mathbf{A}^T$ is orthogonal which means we need to show $(\mathbf{A}^T)^T\mathbf{A}^T = \mathbf{I}$. Considering the LHS of this

$$(\mathbf{A}^T)^T\mathbf{A}^T = \mathbf{A}\mathbf{A}^T \qquad (*)$$

Applying the following proposition of chapter 4:

> Proposition (4.20). **Q** is an orthogonal matrix $\Leftrightarrow$ $\mathbf{Q}^{-1} = \mathbf{Q}^T$.

Since we are given that **A** is orthogonal so $\mathbf{A}^{-1} = \mathbf{A}^T$. Substituting this into (*) yields

$$(\mathbf{A}^T)^T\mathbf{A}^T = \mathbf{A}\mathbf{A}^T = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$$

We have $(\mathbf{A}^T)^T\mathbf{A}^T = \mathbf{I}$ so by Proposition (4.19) we have matrix $\mathbf{A}^T$ is orthogonal.

■

22. Remember an orthogonal matrix **Q** satisfies:

Proposition (4.19). $\mathbf{Q}$ is an orthogonal matrix $\Leftrightarrow$ $\mathbf{Q}^T\mathbf{Q}=\mathbf{I}$

We need to show for the given matrix $\mathbf{Q}=(\mathbf{I}-\mathbf{A})(\mathbf{I}+\mathbf{A})^{-1}$ that $\mathbf{Q}^T\mathbf{Q}=\mathbf{I}$.

$$\begin{aligned}
\mathbf{Q}^T &= \left[(\mathbf{I}-\mathbf{A})(\mathbf{I}+\mathbf{A})^{-1}\right]^T \\
&= \left[(\mathbf{I}+\mathbf{A})^{-1}\right]^T(\mathbf{I}-\mathbf{A})^T && \left[\text{By (1.19) (d) } (\mathbf{AB})^T=\mathbf{B}^T\mathbf{A}^T\right] \\
&= \left[(\mathbf{I}+\mathbf{A})^T\right]^{-1}(\mathbf{I}-\mathbf{A})^T && \left[\text{By (1.29) } \left(\mathbf{X}^T\right)^{-1}=\left(\mathbf{X}^{-1}\right)^T\right] \\
&= \left(\mathbf{I}^T+\mathbf{A}^T\right)^{-1}\left(\mathbf{I}^T-\mathbf{A}^T\right) && \left[\text{By (1.19) (c) } (\mathbf{X}+\mathbf{Y})^T=\mathbf{X}^T+\mathbf{Y}^T\right] \\
&= (\mathbf{I}-\mathbf{A})^{-1}(\mathbf{I}+\mathbf{A}) && \left[\text{Because } \mathbf{A}^T=-\mathbf{A} \text{ (anti-symmetric)}\right]
\end{aligned}$$

We have $\mathbf{Q}^T=(\mathbf{I}-\mathbf{A})^{-1}(\mathbf{I}+\mathbf{A})$ multiplying this by $\mathbf{Q}=(\mathbf{I}-\mathbf{A})(\mathbf{I}+\mathbf{A})^{-1}$ gives:

$$\begin{aligned}
\mathbf{Q}^T\mathbf{Q} &= (\mathbf{I}-\mathbf{A})^{-1}(\mathbf{I}+\mathbf{A})(\mathbf{I}-\mathbf{A})(\mathbf{I}+\mathbf{A})^{-1} \\
&= (\mathbf{I}-\mathbf{A})^{-1}\left(\mathbf{I}-\mathbf{A}^2\right)(\mathbf{I}+\mathbf{A})^{-1} \\
&= \underbrace{(\mathbf{I}-\mathbf{A})^{-1}(\mathbf{I}-\mathbf{A})}_{=\mathbf{I}}\underbrace{(\mathbf{I}+\mathbf{A})(\mathbf{I}+\mathbf{A})^{-1}}_{=\mathbf{I}}=\mathbf{I}\times\mathbf{I}=\mathbf{I}
\end{aligned}$$

Hence $\mathbf{Q}$ is an orthogonal matrix.

■

23. (i) Using the inner product $\langle\mathbf{u},\ \mathbf{v}\rangle=\mathbf{u}^T\mathbf{v}$ we have

$$\begin{aligned}
\langle\mathbf{Ax},\ \mathbf{x}\rangle=(\mathbf{Ax})^T\mathbf{x}=\mathbf{x}^T\mathbf{A}^T\mathbf{x} \\
&= \begin{pmatrix} x & y\end{pmatrix}\begin{pmatrix} a & b \\ -b & c\end{pmatrix}^T\begin{pmatrix} x \\ y\end{pmatrix} \\
&= \begin{pmatrix} x & y\end{pmatrix}\begin{pmatrix} a & -b \\ b & c\end{pmatrix}\begin{pmatrix} x \\ y\end{pmatrix} \\
&= \begin{pmatrix} x & y\end{pmatrix}\begin{pmatrix} ax-by \\ bx+cy\end{pmatrix} \\
&= ax^2-bxy+bxy+cy^2 \\
&= ax^2+cy^2\geq 0 && \left[\text{Because } a>0 \text{ and } c>0\right]
\end{aligned}$$

(ii) By part (i) we have $\langle\mathbf{Ax},\ \mathbf{x}\rangle=ax^2+cy^2$. Therefore

$$ax^2+cy^2=0 \quad\Leftrightarrow\quad x^2=-\frac{c}{a}y^2$$

x and *y* are real because $\mathbf{x}=\begin{pmatrix} x \\ y\end{pmatrix}\in\mathbb{R}^2$ so $x=y=0$. Hence $\mathbf{x}=\mathbf{O}$.

24. We need to find 2 vectors which are orthogonal to both the given vectors **u** and **v**. We can ignore the fraction ½ because this just normalizes the vectors. Let $\mathbf{x} = \begin{pmatrix} x & y & z & w \end{pmatrix}^T$. Then this vector must satisfy

$$\mathbf{u} \cdot \mathbf{x} = 0 \text{ and } \mathbf{v} \cdot \mathbf{x} = 0$$

Need to solve

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = x + y + z + w = 0 \quad \text{and} \quad \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = x + y - z - w = 0$$

Thus writing these simultaneous equations:

$$x + y + z + w = 0 \qquad (\dagger)$$
$$x + y - z - w = 0 \qquad (\dagger\dagger)$$

Adding the two equations $(\dagger)$ and $(\dagger\dagger)$ gives

$$x + y = 0 \quad \Rightarrow \quad x = -y$$

Let $y = 1$ then $x = -1$. The remaining variables *z* and *w* are free so we can enter any real number for these. Let $z = w = 0$ then one vector orthogonal to both the given vectors is

$$\mathbf{x} = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Let $\mathbf{y} = \begin{pmatrix} x' & y' & z' & w' \end{pmatrix}^T$ be our last vector which needs to be orthogonal to all the previous vectors **u**, **v** and **x**:

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} x' \\ y' \\ z' \\ w' \end{pmatrix} = x' + y' + z' + w' = 0, \quad \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} x' \\ y' \\ z' \\ w' \end{pmatrix} = x' + y' - z' - w' = 0, \quad \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} x' \\ y' \\ z' \\ w' \end{pmatrix} = -x' + y' = 0$$

Need to solve these three linear equations

$$x' + y' + z' + w' = 0 \qquad (*)$$
$$x' + y' - z' - w' = 0 \qquad (**)$$
$$-x' + y' \qquad = 0 \quad \Rightarrow \quad y' = x'$$

Substituting $y' = x'$ into the first two equations and adding gives

$$2x' + z' + w' = 0$$
$$\underline{2x' - z' - w' = 0}$$
$$4x' \qquad = 0 \quad \Rightarrow \quad x' = 0$$

Because we have $y' = x'$ so $y' = 0$. Putting this $y' = x' = 0$ into (*) and (**) gives us

$$z' = -w'$$

Let $w' = 1$ then $z' = -w' = -1$. Hence

$$\mathbf{y} = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

The family $\{\mathbf{u},\ \mathbf{v},\ \mathbf{x},\ \mathbf{y}\}$ form an orthogonal basis for $\mathbb{R}^4$. We only need to normalize the vectors **x** and **y** to get an orthonormal basis:

$$\hat{\mathbf{x}} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \text{ and } \hat{\mathbf{y}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

Hence $\{\mathbf{u},\ \mathbf{v},\ \hat{\mathbf{x}},\ \hat{\mathbf{y}}\}$ is an orthonormal basis for $\mathbb{R}^4$.

25. Let $(x \quad y \quad z \quad w)^T$ be the vector which is orthogonal to the vectors in *S*. Remember for vectors to be orthogonal their inner (dot) product need to be zero. We have

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \\ 2 \end{pmatrix} = 0 \quad \Rightarrow \quad x - z + 2w = 0 \qquad (*)$$

Also

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix} = 0 \quad \Rightarrow \quad -x + y + z = 0 \qquad (**)$$

From the first equation (*) we have

$$x = z - 2w$$

Let $z = s$ and $w = t$ then $x = z - 2w = s - 2t$. Using these values in the second equation (**) yields

$$y = x - z = s - 2t - s = -2t$$

Our solution is $x = s - 2t$, $y = -2t$, $z = s$ and $w = t$. In vector form we have

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} s-2t \\ -2t \\ s \\ t \end{pmatrix} = s\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -2 \\ -2 \\ 0 \\ 1 \end{pmatrix}$$

Hence a basis for the subspace which is orthogonal to *S* is

$$B = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ -2 \\ 0 \\ 1 \end{pmatrix} \right\}$$

To find an orthogonal basis we need to apply the Gram Schmidtt Process to these vectors in *B*. The Gram Schimtt Process was described in chapter 4 as follows:

Gram Schmidt Process (4.16).

Let $\mathbf{p}_1 = \mathbf{v}_1$

$$\mathbf{p}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \ \mathbf{p}_1 \rangle}{\|\mathbf{p}_1\|^2} \mathbf{p}_1$$

Let $\mathbf{p}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ then

$$\mathbf{p}_2 = \begin{pmatrix} -2 \\ -2 \\ 0 \\ 1 \end{pmatrix} - \frac{\begin{pmatrix} -2 \\ -2 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}}{\left\| \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\|^2} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \\ 0 \\ 1 \end{pmatrix} - \frac{-2}{2} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2+1 \\ -2+0 \\ 0+1 \\ 1+0 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ 1 \\ 1 \end{pmatrix}$$

Our orthonormal basis for $S^\perp$ is

$$\hat{\mathbf{p}}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \ \hat{\mathbf{p}}_2 = \frac{1}{\sqrt{7}} \begin{pmatrix} -1 \\ -2 \\ 1 \\ 1 \end{pmatrix}$$

26. *Proof*.

(a) Since matrices **A** and **B** are orthogonal so by:

Proposition (4.20). **Q** is an orthogonal matrix $\Leftrightarrow$ $\mathbf{Q}^{-1} = \mathbf{Q}^T$.

We have $\mathbf{A}^{-1} = \mathbf{A}^T$, $\mathbf{B}^{-1} = \mathbf{B}^T$. Required to prove that $(\mathbf{AB})^{-1} = (\mathbf{AB})^T$. Consider the Left Hand Side of this

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1} = \mathbf{B}^T\mathbf{A}^T = (\mathbf{AB})^T$$

Hence by (4.20) the matrix product **AB** is an orthogonal matrix.

■

(b) We are given that matrix **A** is orthogonal so by Proposition (4.20) we have $\mathbf{A}^{-1} = \mathbf{A}^T$. Required to prove that the inverse of matrix $\mathbf{A}^{-1}$ is $\left(\mathbf{A}^{-1}\right)^T$:

$$\left(\mathbf{A}^{-1}\right)^{-1} = \mathbf{A} = \left(\mathbf{A}^T\right)^T = \left(\mathbf{A}^{-1}\right)^T$$

By Proposition (4.20) the matrix $\mathbf{A}^{-1}$ is orthogonal.

■

27. First we check the orthogonality of p_0 and p_1:

$$\langle 1,\ 1-x\rangle = \int_0^\infty (1-x)e^{-x}\mathrm{d}x = 0 \qquad [\text{Using integration by parts}]$$

Hence p_0 and p_1 are orthogonal to each other. Next we check to see if they, p_0 and p_1, have a norm of 1:

$$\|1\| = \langle 1,\ 1\rangle = \int_0^\infty e^{-x}\mathrm{d}x = 1$$

$$\|1-x\| = \langle 1-x,\ 1-x\rangle = \int_0^\infty (1-x)^2 e^{-x}\mathrm{d}x = 1$$

Hence $\{p_0,\ p_1\}$ is an orthonormal set of vectors.

28. The vectors $\dfrac{\mathbf{u}}{\|\mathbf{u}\|},\ \dfrac{\mathbf{v}}{\|\mathbf{v}\|}$ are unit vectors. By Cauchy Schwarz Inequality:

(4.6). Let **u** and **v** be vectors in an inner product space then

$$|\langle \mathbf{u},\ \mathbf{v}\rangle| \le \|\mathbf{u}\|\|\mathbf{v}\|$$

Applying this (4.6) to $\dfrac{\mathbf{u}}{\|\mathbf{u}\|},\ \dfrac{\mathbf{v}}{\|\mathbf{v}\|}$ gives

$$\left|\left\langle \frac{\mathbf{u}}{\|\mathbf{u}\|},\ \frac{\mathbf{v}}{\|\mathbf{v}\|}\right\rangle\right| \le \left\|\frac{\mathbf{u}}{\|\mathbf{u}\|}\right\| \times \left\|\frac{\mathbf{v}}{\|\mathbf{v}\|}\right\| = 1 \qquad \left[\text{Because } \frac{\mathbf{u}}{\|\mathbf{u}\|},\ \frac{\mathbf{v}}{\|\mathbf{v}\|} \text{ are unit vectors}\right]$$

Of course $\left|\left\langle \dfrac{\mathbf{u}}{\|\mathbf{u}\|},\ \dfrac{\mathbf{v}}{\|\mathbf{v}\|}\right\rangle\right| \ge 0$ because we take the modulus which cannot be negative.

When is this zero?

Vectors **u** and **v** are orthogonal.

This completes our required proof of $0 \le \left|\left\langle \dfrac{\mathbf{u}}{\|\mathbf{u}\|},\ \dfrac{\mathbf{v}}{\|\mathbf{v}\|}\right\rangle\right| \le 1$.