Solutions to Problems on Chapter 1 (Only available to tutors.)

1. 524288
2. Yes.
3. Since **A** is a symmetric matrix so $\mathbf{A}^T = \mathbf{A}$ and

$$\left(\mathbf{A}^T\mathbf{A}\right)^n = \left(\mathbf{A}^2\right)^n = \mathbf{A}^{2n}$$

4. Need to find **AB**

(a) $\mathbf{AB} = \begin{pmatrix}1\\7\end{pmatrix}\begin{pmatrix}1 & 2 & 3 & 4\end{pmatrix} = \begin{pmatrix}1 & 2 & 3 & 4\\7 & 14 & 21 & 28\end{pmatrix}$

(b) $\mathbf{AB} = \begin{pmatrix}1\\2\\3\end{pmatrix}\begin{pmatrix}5 & 6\end{pmatrix} = \begin{pmatrix}5 & 6\\10 & 12\\15 & 18\end{pmatrix}$

(c) $\mathbf{AB} = \begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}3\\4\end{pmatrix} = \begin{pmatrix}11\end{pmatrix} = 11$

(d) $\mathbf{AB} = \begin{pmatrix}3\\4\end{pmatrix}\begin{pmatrix}1 & 2\end{pmatrix} = \begin{pmatrix}3 & 6\\4 & 8\end{pmatrix}$

5. We have

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}a_1 & b_1 & c_1 & d_1\\a_2 & b_2 & c_2 & d_2\\a_3 & b_3 & c_3 & d_3\end{pmatrix}\begin{pmatrix}t^3\\t^2\\t\\1\end{pmatrix}$$

6. We have

$$\begin{aligned}\sum_{j=1}^{2}\mathrm{col}_j(\mathbf{A})\mathrm{row}_j(\mathbf{B}) &= \mathrm{col}_1(\mathbf{A})\mathrm{row}_1(\mathbf{B}) + \mathrm{col}_2(\mathbf{A})\mathrm{row}_2(\mathbf{B})\\
&= \begin{pmatrix}a_{11}\\a_{21}\end{pmatrix}\begin{pmatrix}b_{11} & b_{12}\end{pmatrix} + \begin{pmatrix}a_{12}\\a_{22}\end{pmatrix}\begin{pmatrix}b_{21} & b_{22}\end{pmatrix}\\
&= \left(\begin{pmatrix}a_{11}\\a_{21}\end{pmatrix}b_{11} \quad \begin{pmatrix}a_{11}\\a_{21}\end{pmatrix}b_{12}\right) + \left(\begin{pmatrix}a_{12}\\a_{22}\end{pmatrix}b_{21} \quad \begin{pmatrix}a_{12}\\a_{22}\end{pmatrix}b_{22}\right)\\
&= \begin{pmatrix}a_{11}b_{11} & a_{11}b_{12}\\a_{21}b_{11} & a_{21}b_{12}\end{pmatrix} + \begin{pmatrix}a_{12}b_{21} & a_{12}b_{22}\\a_{22}b_{21} & a_{22}b_{22}\end{pmatrix}\\
&= \begin{pmatrix}a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22}\\a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22}\end{pmatrix}\end{aligned}$$

The given formula gives us the matrix multiplication **AB**.

7. The second row of the matrix multiplication **AB** is given by

$$\left(\text{row}_2(\mathbf{A})\times\mathbf{B}\right)=\left(\begin{pmatrix}4 & 5 & 6\end{pmatrix}\begin{pmatrix}1 & 2\\ 3 & 4\\ 5 & 6\end{pmatrix}\right)=\begin{pmatrix}4+15+30 & 8+20+36\end{pmatrix}=\begin{pmatrix}49 & 64\end{pmatrix}$$

8. The fourth column of matrix **B** is highlighted; $\mathbf{B}=\begin{pmatrix}5 & 1 & 1 & 2 & 4 & 5 & 6\\ 3 & 7 & 6 & 9 & 1 & 3 & 5\\ 2 & 5 & 7 & 8 & 6 & 1 & 3\\ 1 & 3 & 5 & 3 & 4 & 6 & 8\end{pmatrix}$.

The fourth column of **AB** is

$$\left[2\times\text{col}_1(\mathbf{A})\right]+\left[9\times\text{col}_2(\mathbf{A})\right]+\left[8\times\text{col}_3(\mathbf{A})\right]+\left[3\times\text{col}_4(\mathbf{A})\right]$$

We have

$$\left[2\begin{pmatrix}1\\5\\9\end{pmatrix}\right]+\left[9\begin{pmatrix}2\\6\\10\end{pmatrix}\right]+\left[8\begin{pmatrix}3\\7\\11\end{pmatrix}\right]+\left[3\begin{pmatrix}4\\8\\12\end{pmatrix}\right]=\begin{pmatrix}2+18+24+12\\ 10+54+56+24\\ 18+90+88+36\end{pmatrix}=\begin{pmatrix}56\\144\\232\end{pmatrix}$$

9. Examples are

$$\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix},\ \begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix},\ \begin{pmatrix}1/2 & 1/2\\ 1/2 & 1/2\end{pmatrix},\ \begin{pmatrix}3/4 & 0 & 1/4 & 0\\ 0 & 1/2 & 1/4 & 1/4\\ 1/3 & 2/3 & 0 & 0\\ 1/7 & 1/2 & 2/7 & 1/14\end{pmatrix}$$

10. (i) Writing out the augmented matrix:

$$\begin{matrix}\text{R}_1\\ \text{R}_2\end{matrix}\left(\begin{array}{ccc|c}2 & 0 & 1 & 0\\ 4 & 1 & 3 & 0\end{array}\right)$$

Carrying out the following row operation:

$$\begin{matrix}\\ \text{R}_1\\ \text{R}_2-2\text{R}_1\end{matrix}\begin{matrix}\begin{matrix}x & y & z\end{matrix}\\ \left(\begin{array}{ccc|c}2 & 0 & 1 & 0\\ 0 & 1 & 1 & 0\end{array}\right)\end{matrix}$$

By examining the bottom row we have

$$y+z=0 \quad\Rightarrow\quad y=-z$$

Let $z=s$ where $s$ is any real number then $y=-s$. From the top row we have

$$2x+z=0 \quad\Rightarrow\quad x=-\frac{z}{2}=-\frac{s}{2}$$

The (homogeneous) solution is $x=-\frac{s}{2}$, $y=-s$ and $z=s$. We can write this homogeneous solution in vector form as:

$$\mathbf{x}=\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}-s/2\\-s\\s\end{pmatrix}=s\begin{pmatrix}-1/2\\-1\\1\end{pmatrix}$$

(ii) Writing out the augmented matrix:

$$\begin{matrix}R_1\\R_2\end{matrix}\left(\begin{array}{ccc|c}2&0&1&2\\4&1&3&1\end{array}\right)$$

Carrying out the following row operation:

$$\begin{matrix}\\R_1\\R_2-2R_1\end{matrix}\begin{matrix}\begin{matrix}x&y&z\end{matrix}\\\left(\begin{array}{ccc|c}2&0&1&2\\0&1&1&-3\end{array}\right)\end{matrix}$$

By examining the bottom row we have

$$y+z=-3 \quad\Rightarrow\quad y=-3-z$$

Let $z=s$ where *s* is any real number then $y=-3-s$. From the top row we have

$$2x+z=2 \quad\Rightarrow\quad 2x=2-z \quad\Rightarrow\quad x=1-\frac{z}{2}=1-\frac{s}{2}$$

The non-homogeneous solution is $x=1-\frac{s}{2}$, $y=-3-s$ and $z=s$. In vector form

$$\mathbf{x}=\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1-s/2\\-3-s\\s\end{pmatrix}=\underbrace{\begin{pmatrix}1\\-3\\0\end{pmatrix}}_{\text{Non-homogeneous}}+\underbrace{s\begin{pmatrix}-1/2\\-1\\1\end{pmatrix}}_{\text{Homogeneous}}$$

(iii) Writing out the augmented matrix:

$$\begin{matrix}R_1\\R_2\end{matrix}\left(\begin{array}{ccc|c}2&0&1&1\\4&1&3&2\end{array}\right)$$

Carrying out the following row operation:

$$\begin{matrix}\\R_1\\R_2-2R_1\end{matrix}\begin{matrix}\begin{matrix}x&y&z\end{matrix}\\\left(\begin{array}{ccc|c}2&0&1&1\\0&1&1&0\end{array}\right)\end{matrix}$$

By examining the bottom row we have

$$y+z=0 \quad\Rightarrow\quad y=-z$$

Let $z=s$ where *s* is any real number then $y=-s$. From the top row we have

$$2x+z=1 \quad\Rightarrow\quad 2x=1-z \quad\Rightarrow\quad x=\frac{1}{2}-\frac{z}{2}=\frac{1}{2}-\frac{s}{2}$$

The non-homogeneous solution is $x=\frac{1}{2}-\frac{s}{2}$, $y=-s$ and $z=s$. In vector form

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1/2 - s/2 \\ -s \\ s \end{pmatrix} = \underbrace{\begin{pmatrix} 1/2 \\ 0 \\ 0 \end{pmatrix}}_{\text{Non-homogeneous}} + \underbrace{s\begin{pmatrix} -1/2 \\ -1 \\ 1 \end{pmatrix}}_{\text{Homogeneous}}$$

11. Let $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 1 & 5 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$ then

$$\mathbf{AB} = \mathbf{BA} = \begin{pmatrix} 5 & 12 \\ 6 & 17 \end{pmatrix}$$

We need to evaluate $(\mathbf{A}+\mathbf{B})^2$:

$$(\mathbf{A}+\mathbf{B})^2 = \mathbf{A}^2 + 2\mathbf{AB} + \mathbf{B}^2 \quad [\text{Because } \mathbf{AB} = \mathbf{BA}]$$

We have

$$\mathbf{A}^2 = \begin{pmatrix} 3 & 2 \\ 1 & 5 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 1 & 5 \end{pmatrix} = \begin{pmatrix} 11 & 16 \\ 8 & 27 \end{pmatrix}$$

$$\mathbf{B}^2 = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 8 \\ 4 & 11 \end{pmatrix}$$

$$(\mathbf{A}+\mathbf{B})^2 = \begin{pmatrix} 11 & 16 \\ 8 & 27 \end{pmatrix} + 2\begin{pmatrix} 5 & 12 \\ 6 & 17 \end{pmatrix} + \begin{pmatrix} 3 & 8 \\ 4 & 11 \end{pmatrix} = \begin{pmatrix} 11+10+3 & 16+24+8 \\ 8+12+4 & 27+34+11 \end{pmatrix} = \begin{pmatrix} 24 & 48 \\ 24 & 72 \end{pmatrix}$$

12. Let $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ then both these matrices are non-invertible but

$$\mathbf{A}+\mathbf{B} = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 3 & 6 \end{pmatrix} \text{ is invertible}$$

13. Writing the augmented matrix for the given linear system:

$$\begin{matrix} \mathrm{R}_1 \\ \mathrm{R}_2 \\ \mathrm{R}_3 \end{matrix} \left(\begin{array}{ccc|c} 1 & 1 & k & 1 \\ 2 & k & -3 & 2 \\ k & 2 & 6 & 0 \end{array}\right)$$

Carrying out the following row operations:

$$\begin{matrix} \mathrm{R}_1 \\ \mathrm{R}_2{*} = \mathrm{R}_2 - 2\mathrm{R}_1 \\ \mathrm{R}_3{*} = \mathrm{R}_3 - k\mathrm{R}_1 \end{matrix} \left(\begin{array}{ccc|c} 1 & 1 & k & 1 \\ 0 & k-2 & -3-2k & 0 \\ 0 & 2-k & 6-k^2 & -k \end{array}\right)$$

Adding the bottom two rows gives

$$\begin{matrix} \mathrm{R}_1 \\ \mathrm{R}_2{*} \\ \mathrm{R}_3{*}+\mathrm{R}_2{*} \end{matrix} \left(\begin{array}{ccc|c} 1 & 1 & k & 1 \\ 0 & k-2 & -3-2k & 0 \\ 0 & 0 & 3-k^2-2k & -k \end{array}\right) \qquad (*)$$

By the bottom row we have

$$\left(3-k^2-2k\right)z=-k \quad \Leftrightarrow \quad \left(k^2+2k-3\right)z=k$$

(a) The given system will have *no* solution if

$$k^2+2k-3=0 \;\Rightarrow\; (k+3)(k-1)=0 \;\Rightarrow\; k=-3,\ 1$$

*Why not?*

Because if $k=-3,\ 1$ then the bottom row is

$$0x+0y+0z=-k=-(-3,\ 1)=9,\ -1$$

This is impossible.

Also if $k=2$ then substituting this into the above augmented matrix (*) gives:

$$\begin{matrix} \text{R}_1 \\ \text{R}_2* \\ \text{R}_3*+\text{R}_2* \end{matrix} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 0 & -7 & 0 \\ 0 & 0 & 3 & -2 \end{array}\right)$$

From the bottom two rows we have

$$3z=-2 \;\Rightarrow\; z=-2/3$$
$$-7z=0 \;\Rightarrow\; z=0$$

Cannot have both these solutions so the system is inconsistent or has no solution if $k=2$.

Hence we have *no* solution if $k=1,\ 2,\ -3$.

(b) By part (a) the linear system will have unique solution provided

$$k\neq 1,\ 2,\ -3$$

(c) For any *k* we cannot have an infinite number of solutions.

14. Since there are 4 leading ones and 4 unknowns so there are no free variables. Hence we have a unique solution which is $x_1=x_2=x_3=x_4=0$.

15. We are given that $a\neq 0,\ b\neq 0$ and $c\neq 0$:

(a) By using row operations we know that we only need to divide the last row by c. Hence

$$\mathbf{A}^{-1}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/c \end{pmatrix}$$

(b) and (c). Similarly we have

$$\mathbf{B}^{-1}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/b & 0 \\ 0 & 0 & 1/c \end{pmatrix} \text{ and } \mathbf{C}^{-1}=\begin{pmatrix} 1/a & 0 & 0 \\ 0 & 1/b & 0 \\ 0 & 0 & 1/c \end{pmatrix}$$

16. (i) Carrying out the matrix multiplication:

$$\mathbf{AB}=\begin{pmatrix}1&0&0\\a&1&0\\b&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&c&1\end{pmatrix}=\begin{pmatrix}1&0&0\\a&1&0\\b&c&1\end{pmatrix}$$

(ii) We need to find the inverse of our result of part (i). We use row operations:

$$\begin{matrix}R_1\\R_2\\R_3\end{matrix}\left(\begin{array}{ccc|ccc}1&0&0&1&0&0\\a&1&0&0&1&0\\b&c&1&0&0&1\end{array}\right)$$

Carrying out the following row operations

$$\begin{matrix}R_1\\R_2^*=R_2-aR_1\\R_3^*=R_3-bR_1\end{matrix}\left(\begin{array}{ccc|ccc}1&0&0&1&0&0\\0&1&0&-a&1&0\\0&c&1&-b&0&1\end{array}\right)$$

Executing

$$\begin{matrix}R_1\\R_2^*\\R_3^*-cR_2^*\end{matrix}\left(\begin{array}{ccc|ccc}1&0&0&1&0&0\\0&1&0&-a&1&0\\0&0&1&-b+ac&-c&1\end{array}\right)$$

Therefore $(\mathbf{AB})^{-1}=\begin{pmatrix}1&0&0\\-a&1&0\\ac-b&-c&1\end{pmatrix}$.

(iii) We have

$$\mathbf{BA}=\begin{pmatrix}1&0&0\\0&1&0\\0&c&1\end{pmatrix}\begin{pmatrix}1&0&0\\a&1&0\\b&0&1\end{pmatrix}=\begin{pmatrix}1&0&0\\a&1&0\\b+ca&c&1\end{pmatrix}$$

(iv) Using row operations to find the inverse:

$$\begin{matrix}R_1\\R_2\\R_3\end{matrix}\left(\begin{array}{ccc|ccc}1&0&0&1&0&0\\a&1&0&0&1&0\\b+ca&c&1&0&0&1\end{array}\right)$$

Executing the following row operations

$$\begin{matrix}R_1\\R_2\\R_3^\dagger=R_3-cR_2\end{matrix}\left(\begin{array}{ccc|ccc}1&0&0&1&0&0\\a&1&0&0&1&0\\b&0&1&0&-c&1\end{array}\right)$$

Carrying out the following two row operations:

$$\begin{matrix}R_1\\R_2-aR_1\\R_3^\dagger-bR_1\end{matrix}\left(\begin{array}{ccc|ccc}1&0&0&1&0&0\\0&1&0&-a&1&0\\0&0&1&-b&-c&1\end{array}\right)$$

Hence $(\mathbf{BA})^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ -b & -c & 1 \end{pmatrix}$.

17. Applying row operations we have:

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \Rightarrow \begin{matrix} R_1 \\ R_2' = R_2 - 4R_1 \\ R_3' = R_3 - 7R_1 \end{matrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{pmatrix}$$

$$\Rightarrow \begin{matrix} R_1 \\ R_2' \\ R_3'' = R_3' - 2R_2' \end{matrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{pmatrix}$$

Since matrix **A** has a row of zeros so it cannot have an inverse.

18. Let us make 2 by 2 matrices **A** and **B** such that $\mathbf{AB} = \mathbf{O}$. One simple example is:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ \mathbf{B} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

19. We have

$$\mathbf{A} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = -1\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = -1(\mathbf{I})$$

$$\mathbf{A}^{2013} = \left[-1(\mathbf{I})\right]^{2013} = -1^{2013}\mathbf{I}^{2013} = -\mathbf{I} = \mathbf{A}$$

20. We need to find

$$\mathbf{A}^T\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ 7 & 8 \\ 9 & 10 \end{pmatrix}^T \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ 7 & 8 \\ 9 & 10 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 5 & 7 & 9 \\ 2 & 4 & 6 & 8 & 10 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ 7 & 8 \\ 9 & 10 \end{pmatrix}$$

$$= \begin{pmatrix} 1+9+25+49+81 & 2+12+30+56+90 \\ 2+12+30+56+90 & 4+16+36+64+100 \end{pmatrix}$$

$$= \begin{pmatrix} 165 & 190 \\ 190 & 220 \end{pmatrix}$$

21. We are given that $\mathbf{E}=\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}$. Evaluating

$$\mathbf{E}^2=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix} \text{ and } \mathbf{E}^3=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

We need to write $\mathbf{A}=\begin{pmatrix}a&b&c\\c&a&b\\b&c&a\end{pmatrix}$ in terms of the matrix **E**:

$$\mathbf{A}=\begin{pmatrix}a&b&c\\c&a&b\\b&c&a\end{pmatrix}=a\mathbf{E}^3+b\mathbf{E}^2+c\mathbf{E}$$

22. The identity matrix **I** satisfies $\mathbf{A}^n-\mathbf{A}=\mathbf{O}$.

23. We are given $\mathbf{A}=\begin{pmatrix}1&2&3\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}1\\2\\3\end{pmatrix}$ and $\mathbf{D}=\begin{pmatrix}1&0&0\\0&2&0\\0&0&3\end{pmatrix}$. Carrying out the matrix multiplication **ADB**:

$$\mathbf{ADB}=\begin{pmatrix}1&2&3\end{pmatrix}\begin{pmatrix}1&0&0\\0&2&0\\0&0&3\end{pmatrix}\begin{pmatrix}1\\2\\3\end{pmatrix}=\begin{pmatrix}1&4&9\end{pmatrix}\begin{pmatrix}1\\2\\3\end{pmatrix}=(1+8+27)=(36)=36$$

24. We need to find $\mathbf{x}^T\mathbf{Ax}$ given $\mathbf{x}=\begin{pmatrix}x\\y\\z\end{pmatrix}$, $\mathbf{A}=\begin{pmatrix}1&2&-6\\2&2&1\\-6&1&3\end{pmatrix}$:

$$\begin{aligned}\mathbf{x}^T\mathbf{Ax}&=\begin{pmatrix}x&y&z\end{pmatrix}\begin{pmatrix}1&2&-6\\2&2&1\\-6&1&3\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}x&y&z\end{pmatrix}\begin{pmatrix}x+2y-6z\\2x+2y+z\\-6x+y+3z\end{pmatrix}\\&=x(x+2y-6z)+y(2x+2y+z)+z(-6x+y+3z)\\&=x^2+2xy-6xz+2xy+2y^2+yz-6xz+yz+3z^2\\&=x^2+4xy+2y^2+2yz+3z^2-12xz\end{aligned}$$

25. Let $\mathbf{A}=\begin{pmatrix}a&b\\c&d\end{pmatrix}$ then

$$\mathbf{A}^2=\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}a&b\\c&d\end{pmatrix}=\begin{pmatrix}a^2+bc&ab+bd\\ac+dc&bc+d^2\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

Equating entries:

$$a^2+bc=0,\ \ (a+d)b=0,\ (a+d)c=0,\ bc+d^2=0$$

From the first and last equations we have

$$bc=-a^2,\quad bc=-d^2 \ \Rightarrow\ a^2=d^2 \ \Rightarrow\ a=\pm d$$

If $a=-d$ then $(a+d)b=0,\ (a+d)c=0$. From $bc=-a^2 \ \Rightarrow\ a=\sqrt{-bc}$. Let $b=1,\ c=-1$ then $a=1$ and $d=-1$. Hence

$$\mathbf{A}^2=\mathbf{O} \text{ with } \mathbf{A}=\begin{pmatrix}1 & 1\\ -1 & -1\end{pmatrix}\neq\mathbf{O}$$

Any 2 by 2 matrix of the form $\mathbf{A}=\begin{pmatrix}x & x\\ -x & -x\end{pmatrix}$ where *x* is any non-zero real number satisfies

$$\mathbf{A}^2=\mathbf{O} \text{ with } \mathbf{A}\neq\mathbf{O}$$

26. *Proof*.
We are given that $\mathbf{AB}=\mathbf{O}$ and **B** is invertible. Right multiply both sides of $\mathbf{AB}=\mathbf{O}$ by $\mathbf{B}^{-1}$:

$$\begin{aligned}(\mathbf{AB})\mathbf{B}^{-1}&=\mathbf{A}(\mathbf{BB}^{-1})\\ &=\mathbf{A}(\mathbf{I})=\mathbf{A}=\mathbf{OB}^{-1}=\mathbf{O}\end{aligned}$$

This completes our proof that $\mathbf{A}=\mathbf{O}$.

■

27. Augmenting the given matrix with the identity gives:

$$\begin{matrix}R_1\\R_2\\R_3\end{matrix}\left(\begin{array}{ccc|ccc}1 & 2 & 3 & 1 & 0 & 0\\ 0 & 4 & 5 & 0 & 1 & 0\\ 0 & 0 & 6 & 0 & 0 & 1\end{array}\right)=(\mathbf{A}\ |\ \mathbf{I})$$

Carrying out the row operation $2R_1-R_2$ gives

$$\begin{matrix}R_1{*}=2R_1-R_2\\R_2\\R_3\end{matrix}\left(\begin{array}{ccc|ccc}2 & 0 & 1 & 2 & -1 & 0\\ 0 & 4 & 5 & 0 & 1 & 0\\ 0 & 0 & 6 & 0 & 0 & 1\end{array}\right)$$

Executing the row operations

$$\begin{matrix}R_1{**}=6R_1{*}-R_3\\R_2{*}=6R_2-5R_3\\R_3\end{matrix}\left(\begin{array}{ccc|ccc}12 & 0 & 0 & 12 & -6 & -1\\ 0 & 24 & 0 & 0 & 6 & -5\\ 0 & 0 & 6 & 0 & 0 & 1\end{array}\right)$$

Dividing each of the rows by 12, 24 and 6 respectively gives

$$\begin{matrix} R_1^{**}/12 \\ R_2^{*}/24 \\ R_3/6 \end{matrix}\left(\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\left|\begin{matrix} 1 & -6/12 & -1/12 \\ 0 & 6/24 & -5/24 \\ 0 & 0 & 1/6 \end{matrix}\right.\right)$$

The inverse matrix is $\mathbf{A}^{-1} = \begin{pmatrix} 1 & -6/12 & -1/12 \\ 0 & 6/24 & -5/24 \\ 0 & 0 & 1/6 \end{pmatrix} = \frac{1}{24}\begin{pmatrix} 24 & -12 & -2 \\ 0 & 6 & -5 \\ 0 & 0 & 4 \end{pmatrix}$.

The solution **x** of the linear system is given by

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} = \frac{1}{24}\begin{pmatrix} 24 & -12 & -2 \\ 0 & 6 & -5 \\ 0 & 0 & 4 \end{pmatrix}\begin{pmatrix} 6 \\ 9 \\ 6 \end{pmatrix} = \frac{1}{24}\begin{pmatrix} 24 \\ 24 \\ 24 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

The solution is $x = y = z = 1$.

28. (a) $\mathbf{A}^2 = \begin{pmatrix} 8 & 10 \\ -6 & -8 \end{pmatrix}\begin{pmatrix} 8 & 10 \\ -6 & -8 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$ and

$$\mathbf{A}^3 = \underbrace{\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}}_{=\mathbf{A}^2}\begin{pmatrix} 8 & 10 \\ -6 & -8 \end{pmatrix} = 4\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 8 & 10 \\ -6 & -8 \end{pmatrix} = 4\begin{pmatrix} 8 & 10 \\ -6 & -8 \end{pmatrix} = 8\begin{pmatrix} 3 & 5 \\ -3 & -4 \end{pmatrix}$$

(b) Use mathematical induction. The three steps of mathematical induction are:

1) Check it for the base case $n = n_0$.

2) Assume it is true for $n = k$.

3) Prove it for $n = k + 1$.

*Proof*.

Let $n = 2m$ be even. Then we need to prove that $\mathbf{A}^{2m} = 2^{2m}\mathbf{I}$.

<u>Step 1</u>

For $m = 1$ we have our result by part (a) because $\mathbf{A}^2 = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} = 2^2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 2^2\mathbf{I}$.

<u>Step 2</u>

Assume that the result is true for $m = k$:

$$\mathbf{A}^{2k} = 2^{2k}\mathbf{I} \qquad (*)$$

<u>Step 3</u>

Required to prove $\mathbf{A}^{2(k+1)} = 2^{2(k+1)}\mathbf{I}$. Examining the Left Hand Side of this gives

$$\mathbf{A}^{2(k+1)} = \mathbf{A}^{2k+2} = \mathbf{A}^{2k}\mathbf{A}^{2} = \underbrace{2^{2k}\mathbf{I}}_{\text{By }(*)}\left(2^{2}\mathbf{I}\right) = 2^{2k+2}\mathbf{I} = 2^{2(k+1)}\mathbf{I}$$

Hence we have our result for even *n*.

Next we need to show the odd *n* result which is $\mathbf{A}^{n} = 2^{n}\begin{pmatrix} 3 & 5 \\ -3 & -4 \end{pmatrix}$.

Let $n = 2m+1$ be odd.

<u>Step 1</u>

For $m = 0$ we have our result because $\mathbf{A}^{1} = 2^{1}\begin{pmatrix} 3 & 5 \\ -3 & -4 \end{pmatrix} = \begin{pmatrix} 8 & 10 \\ -6 & -8 \end{pmatrix}$.

<u>Step 2</u>

We assume that the result is true for $m = k$ :

$$\mathbf{A}^{2k+1} = 2^{2k+1}\begin{pmatrix} 3 & 5 \\ -3 & -4 \end{pmatrix} \quad (**)$$

<u>Step 3</u>

Required to prove $\mathbf{A}^{2(k+1)+1} = 2^{2(k+1)+1}\begin{pmatrix} 3 & 5 \\ -3 & -4 \end{pmatrix}$. Expanding the Left Hand Side:

$$\begin{aligned}\mathbf{A}^{2(k+1)+1} = \mathbf{A}^{2k+3} = \mathbf{A}^{2k+1}\mathbf{A}^{2} &= 2^{2k+1}\begin{pmatrix} 3 & 5 \\ -3 & -4 \end{pmatrix}\mathbf{A}^{2} && [\text{By } (**)] \\ &= 2^{2k+1}\begin{pmatrix} 3 & 5 \\ -3 & -4 \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} && \left[\text{Because by part (a) } \mathbf{A}^{2} = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}\right] \\ &= 2^{2k+1}2^{2}\begin{pmatrix} 3 & 5 \\ -3 & -4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= 2^{2k+1+2}\begin{pmatrix} 3 & 5 \\ -3 & -4 \end{pmatrix} = 2^{2(k+1)+1}\begin{pmatrix} 3 & 5 \\ -3 & -4 \end{pmatrix}\end{aligned}$$

Hence we have our result for odd *n*.

This completes our proof.

■

29. *Proof*

Let **A** and **B** be anti-symmetric matrices. This means that

$$\mathbf{A} = -\mathbf{A}^{T} \quad \text{and} \quad \mathbf{B} = -\mathbf{B}^{T}$$

We need to prove that $(\mathbf{AB} - \mathbf{BA}) = -(\mathbf{AB} - \mathbf{BA})^{T}$. Expanding the right hand side gives

$$
\begin{aligned}
-(\mathbf{AB}-\mathbf{BA})^T &= -\left((\mathbf{AB})^T-(\mathbf{BA})^T\right) && \left[\text{By (1.19) (c) } (\mathbf{A}+\mathbf{B})^T=\mathbf{A}^T+\mathbf{B}^T\right]\\
&= -\left(\mathbf{B}^T\mathbf{A}^T-\mathbf{A}^T\mathbf{B}^T\right) && \left[\text{By (1.19) (d) } (\mathbf{AB})^T=\mathbf{B}^T\mathbf{A}^T\right]\\
&= -\left(-\mathbf{B}(-\mathbf{A})-(-\mathbf{A})(-\mathbf{B})\right) && \left[\text{Because } \mathbf{A}=-\mathbf{A}^T \text{ and } \mathbf{B}=-\mathbf{B}^T\right]\\
&= \mathbf{B}(-\mathbf{A})+\mathbf{AB}=\mathbf{AB}-\mathbf{BA}
\end{aligned}
$$

Hence $\mathbf{AB}-\mathbf{BA}$ is also an anti-symmetric matrix. This completes our proof.

■

30. *Proof.*

Let **AB** be invertible then there is a matrix **X** such that

$$\mathbf{AB}(\mathbf{X})=\mathbf{I}$$

Using the rules of matrix algebra we have

$$\mathbf{AB}(\mathbf{X})=\mathbf{A}(\mathbf{BX})=\mathbf{I}$$

By

> Definition (1.23). A square matrix **A** is said to be **invertible** if there is a matrix **Y** of the same size such that $\mathbf{AY}=\mathbf{YA}=\mathbf{I}$.

Matrix **A** is invertible.

Similarly we have

$$\mathbf{X}(\mathbf{AB})=(\mathbf{XA})\mathbf{B}=\mathbf{I}$$

Therefore matrix **B** is invertible.

■

31. Let $\mathbf{A}=\begin{pmatrix} a_{11} & \cdots & a_{1n}\\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn}\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix} b_{11} & \cdots & b_{1n}\\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nn}\end{pmatrix}$. We use the following formula:

> (1.15) $\qquad (\mathbf{AB})_{ij}=(a_{i1}b_{1j})+(a_{i2}b_{2j})+(a_{i3}b_{3j})+\cdots+(a_{in}b_{nj})$

We only need the leading diagonal elements for the trace of a matrix $\mathbf{AB}$ which are given by $(\mathbf{AB})_{11}$, $(\mathbf{AB})_{22}$, $(\mathbf{AB})_{33}$, $\cdots$, $(\mathbf{AB})_{nn}$. Putting $i=j$ into formula (1.15) gives

$$
\begin{aligned}
(\mathbf{AB})_{11} &= (a_{11}b_{11})+(a_{12}b_{21})+(a_{13}b_{31})+\cdots+(a_{1n}b_{n1})\\
(\mathbf{AB})_{22} &= (a_{21}b_{12})+(a_{22}b_{22})+(a_{23}b_{32})+\cdots+(a_{2n}b_{n2})\\
(\mathbf{AB})_{33} &= (a_{31}b_{13})+(a_{32}b_{23})+(a_{33}b_{33})+\cdots+(a_{3n}b_{n3})\\
\vdots \quad &\qquad \vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots\\
(\mathbf{AB})_{nn} &= (a_{n1}b_{1n})+(a_{n2}b_{2n})+(a_{n3}b_{3n})+\cdots+(a_{nn}b_{nn})
\end{aligned}
$$

First column

Let us add the above because we want to find the trace. Adding the first column in the above gives

$$\left(a_{11}b_{11}\right)+\left(a_{21}b_{12}\right)+\left(a_{31}b_{13}\right)+\cdots+\left(a_{n1}b_{1n}\right) \qquad (*)$$

Adding the second column gives

$$\left(a_{12}b_{21}\right)+\left(a_{22}b_{22}\right)+\left(a_{32}b_{23}\right)+\cdots+\left(a_{n2}b_{2n}\right) \qquad (**)$$

Similarly applying the following to the matrix multiplication the other way round:

$$(\mathbf{BA})_{ij}=\left(b_{i1}a_{1j}\right)+\left(b_{i2}a_{2j}\right)+\left(b_{i3}a_{3j}\right)+\cdots+\left(b_{in}a_{rn}\right)$$

We have

$$(\mathbf{BA})_{11}=\left(b_{11}a_{11}\right)+\left(b_{12}a_{21}\right)+\left(b_{13}a_{31}\right)+\cdots+\left(b_{1n}a_{n1}\right)$$

*What do you notice about this result?*
It is identical to the sum in (*).
Find the second leading diagonal element in **BA**:

$$(\mathbf{BA})_{22}=\left(b_{21}a_{12}\right)+\left(b_{22}a_{22}\right)+\left(b_{23}a_{32}\right)+\cdots+\left(b_{2n}a_{n2}\right)$$

This sum is equal to the sum in (**).
Similarly

$$\begin{array}{l}(\mathbf{BA})_{33}=\left(b_{31}a_{13}\right)+\left(b_{32}a_{23}\right)+\left(b_{33}a_{33}\right)+\cdots+\left(b_{3n}a_{n3}\right)=\text{sum of third column}\\ \quad\vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots\\ (\mathbf{BA})_{nn}=\left(b_{n1}a_{1n}\right)+\left(b_{n2}a_{2n}\right)+\left(b_{n3}a_{3n}\right)+\cdots+\left(b_{nn}a_{nn}\right)=\text{sum of last column}\end{array}$$

Therefore we have

$$(\mathbf{AB})_{11}+(\mathbf{AB})_{22}+(\mathbf{AB})_{33}+\cdots+(\mathbf{AB})_{nn}=(\mathbf{BA})_{11}+(\mathbf{BA})_{22}+(\mathbf{BA})_{33}+\cdots+(\mathbf{BA})_{nn}$$

Hence we have required result $trace(\mathbf{AB})=trace(\mathbf{BA})$.

■

32. (a) A 2 by 2 example of a nilpotent matrix is $\mathbf{A}=\begin{pmatrix}0&1\\0&0\end{pmatrix}$ then $\mathbf{A}^2=\mathbf{O}$. We could replace the entry 1 by any non-zero number and we would still have an nilpotent matrix; $\mathbf{A}=\begin{pmatrix}0&a\\0&0\end{pmatrix}$ where *a* is a non-zero real number.

A 3 by 3 nilpotent matrix is $\mathbf{A}=\begin{pmatrix}0&1&2\\0&0&3\\0&0&0\end{pmatrix}$ because $\mathbf{A}^2=\begin{pmatrix}0&0&3\\0&0&0\\0&0&0\end{pmatrix}$ and $\mathbf{A}^3=\mathbf{O}$.

Actually any triangular matrix with zeros on the leading diagonal will do.
(b) *Proof*.
Let matrices **A** and **B** be nilpotent matrices such that

$$\mathbf{A}^k=\mathbf{O} \quad \text{and} \quad \mathbf{B}^m=\mathbf{O} \quad \text{where } k \text{ and } m \text{ are positive integers}$$

Without loss of generality assume that $m \geq k$. Then

$$(\mathbf{AB})^m = \mathbf{A}^m\mathbf{B}^m = \mathbf{A}^k\mathbf{A}^{m-k}\mathbf{B}^m = \mathbf{O}\mathbf{A}^{m-k}\mathbf{O} = \mathbf{O}$$

Hence **AB** is a nilpotent matrix.

■

(c) Let $\mathbf{A} = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & 3 & 0 \end{pmatrix}$ then both **A** and **B** are nilpotent matrices. However

$$\mathbf{A}+\mathbf{B} = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & 3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 2 & 3 & 0 \end{pmatrix}$$

The matrix $\mathbf{A}+\mathbf{B}$ is *not* a nilpotent matrix.

33. By solution to question 32(a) we know $\mathbf{A} = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}$ is a nilpotent matrix. By using the results of question 32(a) we have

$$\begin{aligned} e^{\mathbf{A}} &= \mathbf{I} + \mathbf{A} + \frac{1}{2!}\mathbf{A}^2 + \frac{1}{3!}\mathbf{A}^3 + \cdots + \frac{1}{n!}\mathbf{A}^n + \cdots \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \cdots + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

34. *Proof*. See solution to question 30 with $\mathbf{B} = \mathbf{A}$.

35. Consider the matrix $(\mathbf{A}+\mathbf{B})^2$:

$$\begin{aligned} (\mathbf{A}+\mathbf{B})^2 &= \mathbf{A}^2 + \underbrace{\mathbf{AB}+\mathbf{BA}}_{=\mathbf{O}} + \mathbf{B}^2 \\ &= \mathbf{A}^2 + \mathbf{B}^2 = \mathbf{I} + \mathbf{I} = 2\mathbf{I} \end{aligned}$$

We have $(\mathbf{A}+\mathbf{B})^2 = 2\mathbf{I}$ and dividing both sides by 2 gives

$$\frac{1}{2}(\mathbf{A}+\mathbf{B})^2 = (\mathbf{A}+\mathbf{B})\left[\frac{1}{2}(\mathbf{A}+\mathbf{B})\right] = \mathbf{I}$$

Therefore the inverse of matrix $(\mathbf{A}+\mathbf{B})$ must be

$$(\mathbf{A}+\mathbf{B})^{-1} = \frac{1}{2}(\mathbf{A}+\mathbf{B})$$

36. (i) Carrying out the matrix multiplication gives

$$\mathbf{AB}=\begin{pmatrix}1&2&3&4\\2&3&4&0\\3&4&0&0\\4&0&0&0\end{pmatrix}\begin{pmatrix}0&0&0&64\\0&0&64&-48\\0&64&-48&4\\64&-48&4&5\end{pmatrix}=\begin{pmatrix}256&0&0&0\\0&256&0&0\\0&0&256&0\\0&0&0&256\end{pmatrix}=256\mathbf{I}$$

(ii) By using the result of part (i) we have

$$\mathbf{A}^{-1}=\frac{1}{256}\mathbf{B}=\frac{1}{256}\begin{pmatrix}0&0&0&64\\0&0&64&-48\\0&64&-48&4\\64&-48&4&5\end{pmatrix}$$

(iii) Similarly we have

$$\mathbf{B}^{-1}=\frac{1}{256}\mathbf{A}=\frac{1}{256}\begin{pmatrix}1&2&3&4\\2&3&4&0\\3&4&0&0\\4&0&0&0\end{pmatrix}$$

37. Since matrix multiplication is *associative* so we have

$$\mathbf{A}(\mathbf{BC})=(\mathbf{AB})\mathbf{C}$$

The Right Hand Side multiplication $(\mathbf{AB})\mathbf{C}$ is a lot easier than the Left Hand Side $\mathbf{A}(\mathbf{BC})$:

$$\begin{aligned}\mathbf{A}(\mathbf{BC})&=(\mathbf{AB})\mathbf{C}\\&=\left[\begin{pmatrix}1&2\\3&4\end{pmatrix}\begin{pmatrix}1&2\\2&1\end{pmatrix}\right]\begin{pmatrix}1&2&3&4&5\\6&7&8&9&10\end{pmatrix}\\&=\begin{pmatrix}5&4\\11&10\end{pmatrix}\begin{pmatrix}1&2&3&4&5\\6&7&8&9&10\end{pmatrix}=\begin{pmatrix}29&38&47&56&65\\71&92&113&134&155\end{pmatrix}\end{aligned}$$

38. We are given that **A** is a *n* by 1 matrix. From the notes on chapter 1 we have

> In general if **A** is a $m\times r$ (*m* rows by *r* columns) matrix and **B** is a $r\times n$ (*r* rows by *n* columns) matrix then the multiplication $\mathbf{AB}$ results in a $m\times n$ matrix.

Applying this we have that matrix $\mathbf{AA}^T$ is an *n* by *n* matrix. We can also write out the entries of the matrices as follows:

Let $\mathbf{A}=\begin{pmatrix}a_{11}\\a_{21}\\\vdots\\a_{n1}\end{pmatrix}$. Then

$$\mathbf{AA}^T = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix} \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{n1} \end{pmatrix} = \left( a_{11}\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix} \quad a_{21}\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix} \quad \cdots \quad a_{n1}\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix} \right)$$

$$= \begin{pmatrix} {a_{11}}^2 & a_{21}a_{11} & \cdots & a_{n1}a_{11} \\ a_{11}a_{21} & {a_{21}}^2 & \cdots & a_{n1}a_{21} \\ \vdots & \vdots & \ddots & \\ a_{11}a_{n1} & a_{21}a_{n1} & & {a_{n1}}^2 \end{pmatrix}$$

Note that $\mathbf{AA}^T$ is an *n* by *n* matrix. To show that this matrix is symmetrical we need to prove $\left(\mathbf{AA}^T\right)^T = \mathbf{AA}^T$:

$$\left(\mathbf{AA}^T\right)^T = \left(\mathbf{A}^T\right)^T \mathbf{A}^T = \mathbf{AA}^T$$

Hence $\mathbf{AA}^T$ is a n by n symmetric matrix.

■

39. $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$ are symmetric matrices. However

$$\mathbf{AB} = \begin{pmatrix} 1 & 8 \\ 2 & 12 \end{pmatrix} \text{is not symmetric}$$

40. Let $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$ then

$$\mathbf{A}+\mathbf{I} = \begin{pmatrix} a_{11}+1 & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}+1 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}+1 \end{pmatrix}$$

Provided that the entries on the leading diagonal of this matrix $\mathbf{A}+\mathbf{I}$ are *not* zero, that is $a_{ii} \neq -1$, then the reduced row echelon form is the identity matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

By the following result of chapter 1:

Theorem (1.35). Let $\mathbf{A}$ be an *n* by *n* matrix, then the following are equivalent:

(a) The matrix **A** is invertible (non-singular).

(b) The reduced row echelon form of the matrix **A** is the identity matrix **I**.

The matrix $\mathbf{A}+\mathbf{I}$ is invertible.

■

41. We use mathematical induction to prove the given result.

Step 1: The result is clearly true for $n=1$:

$$\mathbf{A}^1 = \mathbf{A} = \begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix} = \begin{pmatrix} a^1 & 0 \\ 0 & 1/a^1 \end{pmatrix} \qquad (*)$$

Step 2: Assume the result is true for $n=k$:

$$\mathbf{A}^k = \begin{pmatrix} a^k & 0 \\ 0 & 1/a^k \end{pmatrix} \qquad (\dagger)$$

Step 3: Required to prove this result for $n=k+1$:

$$\mathbf{A}^{k+1} = \mathbf{A}^k\mathbf{A} = \underbrace{\begin{pmatrix} a^k & 0 \\ 0 & 1/a^k \end{pmatrix}}_{\text{By } (\dagger)} \underbrace{\begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix}}_{\text{By } (*)} = \begin{pmatrix} a^{k+1} & 0 \\ 0 & 1/a^{k+1} \end{pmatrix}$$

By mathematical induction we have our required result, $\mathbf{A}^n = \begin{pmatrix} a^n & 0 \\ 0 & 1/a^n \end{pmatrix}$.

The inverse of this $\mathbf{A}^n = \begin{pmatrix} a^n & 0 \\ 0 & 1/a^n \end{pmatrix}$ (using row operations) is given by

$$\left(\mathbf{A}^n\right)^{-1} = \begin{pmatrix} 1/a^n & 0 \\ 0 & a^n \end{pmatrix}$$

42. Let $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. Then

$$\left(\mathbf{A}+\mathbf{B}\right)^{-1} = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\right]^{-1} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{3}\mathbf{I}$$

Additionally we have

$$\mathbf{A}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I} \text{ and } \mathbf{B}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{2}\mathbf{I}$$

Therefore $\mathbf{A}^{-1}+\mathbf{B}^{-1} = \mathbf{I}+\frac{1}{2}\mathbf{I} = \frac{3}{2}\mathbf{I}$. Hence $\left(\mathbf{A}+\mathbf{B}\right)^{-1} \neq \mathbf{A}^{-1}+\mathbf{B}^{-1}$.

43. Since $\mathbf{A}^2$ is invertible we have a matrix **B** such that

$$\mathbf{A}^2\mathbf{B} = \mathbf{I}$$

Using the rules of associativity we have

$$\mathbf{A}^2\mathbf{B} = \mathbf{A}\left(\mathbf{AB}\right) = \mathbf{I}$$

Hence **AB** is the inverse of matrix **A** which means **A** is invertible.

■

44. Re-arranging the given equation we have

$$\mathbf{A}^3+\mathbf{A}^2+\mathbf{A}=10\mathbf{I}$$
$$\mathbf{A}\left(\mathbf{A}^2+\mathbf{A}+\mathbf{I}\right)=10\mathbf{I}$$
$$\mathbf{A}\left[\frac{1}{10}\left(\mathbf{A}^2+\mathbf{A}+\mathbf{I}\right)\right]=\mathbf{I}$$

Since we have **A** times another matrix gives the identity matrix so matrix **A** is invertible. The inverse of **A** is given by

$$\mathbf{A}^{-1}=\frac{1}{10}\left(\mathbf{A}^2+\mathbf{A}+\mathbf{I}\right)$$

45. Let $\mathbf{A}=\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$. We apply the following to $\mathbf{Ax}=\mathbf{b}$ where $\mathbf{x}=\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\mathbf{b}=\begin{pmatrix} 3 \\ 4 \end{pmatrix}$:

(1.10) $$\mathbf{Ax}=\begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}=x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n$$

We have

$$\mathbf{Ax}=\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}=1\begin{pmatrix} a \\ d \end{pmatrix}+2\begin{pmatrix} b \\ e \end{pmatrix}+3\begin{pmatrix} c \\ f \end{pmatrix}=\begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

The equations are

$$a+2b+3c=3 \quad\Rightarrow\quad a=3-2b-3c$$
$$d+2e+3f=4 \quad\Rightarrow\quad d=4-2e-3f$$

Let $b=c=1$ then $a=3-2b-3c=3-2-3=-2$. Let $e=f=-1$ then $d=4-2e-3f=4+2+3=9$. Hence our matrix **A** is given by $\begin{pmatrix} -2 & 1 & 1 \\ 9 & -1 & -1 \end{pmatrix}$.

46. Error in the first line. Remember $(\mathbf{BA})^{-1}=\mathbf{A}^{-1}\mathbf{B}^{-1}$ but $(\mathbf{BA})^{-1}\neq\mathbf{B}^{-1}\mathbf{A}^{-1}$ [Not Equal].

47. We have used this $(\mathbf{I}+\mathbf{A})^{-1}=\mathbf{I}+\mathbf{A}^{-1}$ which is false. See question 42.

48. We need to solve

$$x\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}+y\begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}=\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

A solution is $x=2,\ y=-1$. Hence the linear combination is

$$2\mathbf{v}-\mathbf{w}=\mathbf{u}$$

49. (i) Carrying out the matrix multiplication:

$$\mathbf{AB}=\begin{pmatrix}1&2&3\\1&1&2\\0&1&2\end{pmatrix}\begin{pmatrix}0&1&-1\\2&-2&-1\\-1&1&1\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}=\mathbf{I}$$

(ii) Since by part (i) $\mathbf{AB}=\mathbf{I}$ so $\mathbf{A}^{-1}=\mathbf{B}$. Using the following result of chapter 1:

> Proposition (1.29). Let **A** be an invertible (non-singular) matrix then
>
> $$\left(\mathbf{A}^T\right)^{-1}=\left(\mathbf{A}^{-1}\right)^T$$

We have

$$\left(\mathbf{A}^T\right)^{-1}=\left(\mathbf{A}^{-1}\right)^T=\mathbf{B}^T=\begin{pmatrix}0&1&-1\\2&-2&-1\\-1&1&1\end{pmatrix}^T=\begin{pmatrix}0&2&-1\\1&-2&1\\-1&-1&1\end{pmatrix}$$

50. Substituting the vector $\mathbf{x}=k_1\mathbf{x}_1+k\ \mathbf{x}_2+\cdots+k_n\mathbf{x}_n$ into $\mathbf{Ax}$ gives

$$\begin{aligned}\mathbf{Ax}&=\mathbf{A}\left(k_1\mathbf{x}_1+k_2\mathbf{x}_2+\cdots+k_n\mathbf{x}_n\right)\\&=k_1\left(\mathbf{Ax}_1\right)+k_2\left(\mathbf{Ax}_2\right)+\cdots+k_n\left(\mathbf{Ax}_n\right)\\&=k_1\left(\mathbf{0}\right)+k_2\left(\mathbf{0}\right)+\cdots+k_n\left(\mathbf{0}\right)\qquad\left[\text{Because }\mathbf{x}_1,\ \mathbf{x}_2,\ \cdots,\ \mathbf{x}_n\ \text{ are solutions to }\mathbf{Ax}=\mathbf{0}\right]\\&=\mathbf{0}\end{aligned}$$

Hence the given linear combination is a solution.

51. Using matrix operations we have

$$\begin{aligned}\mathbf{A}\left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T&=\mathbf{A}\left[\mathbf{A}^{-1}\left(\mathbf{A}^T\right)^{-1}\right]\mathbf{A}^T\\&=\left(\mathbf{AA}^{-1}\right)\left(\mathbf{A}^{-1}\right)^T\mathbf{A}^T=\left(\mathbf{AA}^{-1}\right)\left(\mathbf{AA}^{-1}\right)^T=\mathbf{I}\times\mathbf{I}=\mathbf{I}\end{aligned}$$

Substituting this into $\mathbf{I}-\mathbf{A}\left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T$ gives

$$\mathbf{I}-\mathbf{I}=\mathbf{O}$$

Re-arranging the given result $\mathbf{I}-\mathbf{A}\left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T=\mathbf{O}$:

$$\mathbf{A}\left[\left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\right]=\mathbf{I}$$

Hence the inverse of matrix **A** is $\left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T$.

52. The error occurs at the line highlighted:

$$\mathbf{X} = \mathbf{AX} + \mathbf{Y}$$
$$\Rightarrow \quad \mathbf{X} - \mathbf{AX} = \mathbf{Y}$$
$$\Rightarrow \quad \mathbf{X}(\mathbf{I} - \mathbf{A}) = \mathbf{Y} \leftarrow \text{Error here}$$
$$\Rightarrow \quad \mathbf{X} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{Y}$$

*What is wrong with this line?*

Remember matrix multiplication is *not* commutative so $\mathbf{AX} \neq \mathbf{XA}$ which means that we cannot factorise matrix **X** out on the LHS.

53. There is *no* error.

54. (i) We use mathematical induction to prove the given result $\mathbf{A}^m = \mathbf{PD}^m\mathbf{P}^{-1}$.

*Proof*.

Clearly the result is true for $m = 1$ because we are given $\mathbf{A} = \mathbf{PDP}^{-1}$.

Assume the result is true for $m = k$:

$$\mathbf{A}^k = \mathbf{PD}^k\mathbf{P}^{-1} \qquad (*)$$

Required to prove the result for $m = k+1$, that is we need to show

$$\mathbf{A}^{k+1} = \mathbf{PD}^{k+1}\mathbf{P}^{-1}$$

Let us expand the LHS:

$$\mathbf{A}^{k+1} = \mathbf{A}^k\mathbf{A} = \underbrace{\mathbf{PD}^k\mathbf{P}^{-1}}_{\text{By }(*)}\underbrace{\mathbf{PDP}^{-1}}_{\text{Given}} = \mathbf{PD}^k\underbrace{\left(\mathbf{P}^{-1}\mathbf{P}\right)}_{=\mathbf{I}}\mathbf{DP}^{-1} = \mathbf{PD}^k\mathbf{DP}^{-1} = \mathbf{PD}^{k+1}\mathbf{P}^{-1}$$

Hence we have our result.

■

(ii) Again we can use mathematical induction to prove this part.

*Proof*.

Clearly $\mathbf{D}^1 = \begin{pmatrix} a^1 & 0 \\ 0 & b^1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \mathbf{D}$. The result is correct for $m = 1$.

Assume the result is true for $m = k$:

$$\mathbf{D}^k = \begin{pmatrix} a^k & 0 \\ 0 & b^k \end{pmatrix} \qquad (*)$$

Consider the matrix $\mathbf{D}^{k+1}$:

$$\mathbf{D}^{k+1} = \mathbf{D}^k\mathbf{D} = \underbrace{\begin{pmatrix} a^k & 0 \\ 0 & b^k \end{pmatrix}}_{\text{By }(*)}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} a^{k+1} & 0 \\ 0 & b^{k+1} \end{pmatrix}$$

Hence we have our required result.

■

(iii) We need to show $\mathbf{A} = \mathbf{PDP}^{-1}$ for

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ -9 & -8 \end{pmatrix}, \ \mathbf{D} = \begin{pmatrix} -5 & 0 \\ 0 & -2 \end{pmatrix} \text{ and } \mathbf{P} = \begin{pmatrix} -1 & -2 \\ 3 & 3 \end{pmatrix}$$

Finding $\mathbf{P}^{-1}$ by row operations:

$$\left(\mathbf{A}\ \middle|\ \mathbf{I}\right)=\left(\begin{array}{cc|cc}-1 & -2 & 1 & 0\\ 3 & 3 & 0 & 1\end{array}\right)\begin{array}{l}R_1\\ R_2\end{array}$$

Executing the following row operation:

$$\begin{array}{r}R_1\\ R_2^*=R_2+3R_1\end{array}\left(\begin{array}{cc|cc}-1 & -2 & 1 & 0\\ 0 & -3 & 3 & 1\end{array}\right)$$

Carrying out the row operation

$$\begin{array}{r}R_1-\frac{2}{3}R_2^*\\ R_2^*\end{array}\left(\begin{array}{cc|cc}-1 & 0 & -1 & -2/3\\ 0 & -3 & 3 & 1\end{array}\right)$$

Multiplying the bottom row by $-\frac{1}{3}$ and top row by $-1$ gives the inverse of matrix **P** on the RHS:

$$\mathbf{P}^{-1}=\begin{pmatrix}1 & 2/3\\ -1 & -1/3\end{pmatrix}$$

Therefore

$$\mathbf{PDP}^{-1}=\begin{pmatrix}-1 & -2\\ 3 & 3\end{pmatrix}\begin{pmatrix}-5 & 0\\ 0 & -2\end{pmatrix}\begin{pmatrix}1 & 2/3\\ -1 & -1/3\end{pmatrix}$$

$$=\frac{1}{3}\begin{pmatrix}5 & 4\\ -15 & -6\end{pmatrix}\begin{pmatrix}3 & 2\\ -3 & -1\end{pmatrix}$$

$$=\frac{1}{3}\begin{pmatrix}3 & 6\\ -27 & -24\end{pmatrix}=\begin{pmatrix}1 & 2\\ -9 & -8\end{pmatrix}=\mathbf{A}$$

Using the result of part (i) we have $\mathbf{A}^8=\mathbf{PD}^8\mathbf{P}^{-1}$ and by part (ii) we have

$$\mathbf{D}^8=\begin{pmatrix}-5^8 & 0\\ 0 & -2^8\end{pmatrix}=\begin{pmatrix}390625 & 0\\ 0 & 256\end{pmatrix}$$

Substituting $\mathbf{P}=\begin{pmatrix}-1 & -2\\ 3 & 3\end{pmatrix}$, $\mathbf{D}^8=\begin{pmatrix}390625 & 0\\ 0 & 256\end{pmatrix}$ and $\mathbf{P}^{-1}=\frac{1}{3}\begin{pmatrix}3 & 2\\ -3 & -1\end{pmatrix}$ into $\mathbf{A}^8=\mathbf{PD}^8\mathbf{P}^{-1}$ gives

$$\mathbf{A}^8=\begin{pmatrix}-1 & -2\\ 3 & 3\end{pmatrix}\begin{pmatrix}390625 & 0\\ 0 & 256\end{pmatrix}\frac{1}{3}\begin{pmatrix}3 & 2\\ -3 & -1\end{pmatrix}$$

$$=\frac{1}{3}\begin{pmatrix}-390\,625 & -512\\ 1171875 & 768\end{pmatrix}\begin{pmatrix}3 & 2\\ -3 & -1\end{pmatrix}$$

$$=\frac{1}{3}\begin{pmatrix}-1170339 & -780738\\ 3513321 & 2342982\end{pmatrix}=\begin{pmatrix}-390113 & -260246\\ 1171107 & 780994\end{pmatrix}$$

55. (a) True.

*Proof*. Let $\mathbf{x}=\mathbf{0}$ then this is a solution to $\mathbf{Ax}=\mathbf{0}$.

(b) False. A counter example is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

The solution of this system is $x = 3,\ \ y = 2,\ z = s,\ \ w = t$ where *s* and *t* are any real numbers.

(c) False. A counter example is

$$\begin{pmatrix} 2 & 3 \\ 5 & 6 \\ -4 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 11 \\ -10 \end{pmatrix}$$

Has the unique solution $x = y = 1$.

(d) False. A counter example is

$$\begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 12 \end{pmatrix}$$

Writing this as an augmented matrix and carrying out row operations:

$$\begin{matrix} R_1 \\ R_2 \end{matrix}\left(\begin{array}{cc|c} 2 & 3 & 4 \\ 4 & 6 & 8 \end{array}\right) \quad \Longrightarrow \quad \begin{matrix} R_1 \\ R_2 - 2R_1 \end{matrix}\left(\begin{array}{cc|c} 2 & 3 & 4 \\ 0 & 0 & 0 \end{array}\right)$$

From the top row we have

$$2x + 3y = 4 \quad \Rightarrow \quad x = \frac{4 - 3y}{2}$$

Let $y = s$ where *s* is any real number then $x = \dfrac{4 - 3s}{2}$.

This linear system has an infinite number of solutions.

(e) True because of the following proposition of chapter 1:

> Proposition (1.32). Let a consistent non-homogeneous linear system $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{b} \neq \mathbf{O}$ be row equivalent to the augmented matrix $(\mathbf{R} \mid \mathbf{b}')$ where **R** is in reduced row echelon form and there be *n* unknowns and *r* non-zero rows in **R**.
>
> If $r < n$ then the linear system $\mathbf{Ax} = \mathbf{b}$ has an infinite number of solutions.

Proof.

The given linear system $\mathbf{Ax} = \mathbf{b}$ is consistent because we have a solution. The number of non-zero rows is less than *n* and $n < k$ where *k* is the number of unknowns so by the above proposition $\mathbf{Ax} = \mathbf{b}$ has an infinite number of solutions.

■

56. (i) Writing the given equations $m + c = 1.7,\ \ 2m + c = 2.6,\ \ 3m + c = 4,\ \ 4m + c = 5$ in matrix form:

$$\mathbf{Ax}=\mathbf{b} \text{ where } \mathbf{A}=\begin{pmatrix}1&1\\2&1\\3&1\\4&1\end{pmatrix},\ \mathbf{x}=\begin{pmatrix}m\\c\end{pmatrix} \text{ and } \mathbf{b}=\begin{pmatrix}1.7\\2.6\\4\\5\end{pmatrix}$$

(ii) We have

$$\mathbf{A}^T\mathbf{A}=\begin{pmatrix}1&2&3&4\\1&1&1&1\end{pmatrix}\begin{pmatrix}1&1\\2&1\\3&1\\4&1\end{pmatrix}=\begin{pmatrix}30&10\\10&4\end{pmatrix}$$

The inverse of this is given by

$$\left(\mathbf{A}^T\mathbf{A}\right)^{-1}=\frac{1}{20}\begin{pmatrix}4&-10\\-10&30\end{pmatrix}$$

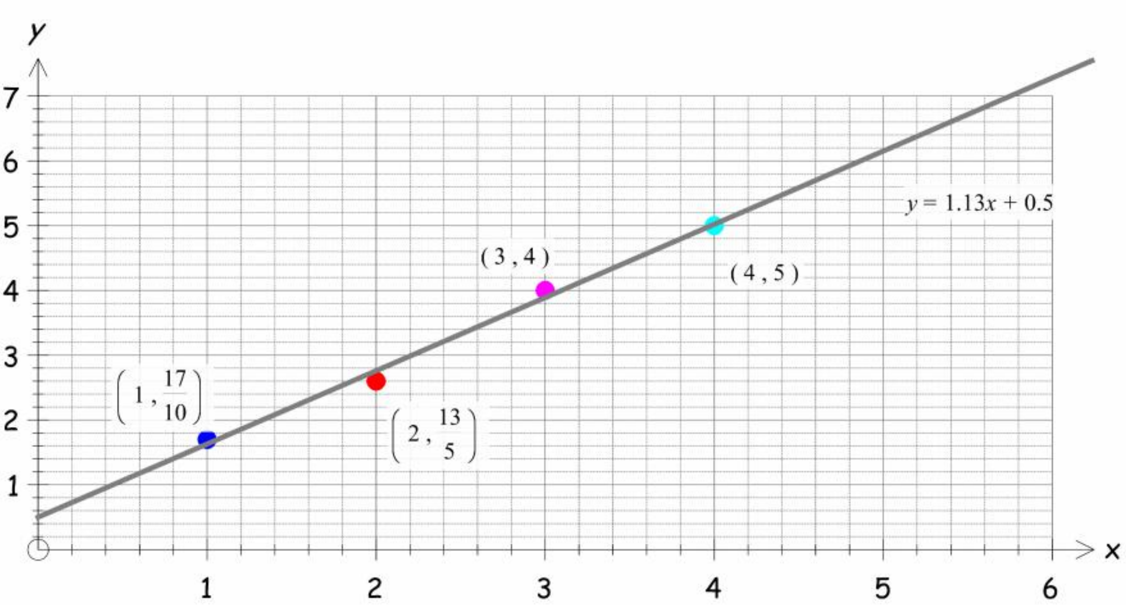
Evaluating $\left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T$ gives

$$\left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T=\frac{1}{20}\begin{pmatrix}4&-10\\-10&30\end{pmatrix}\begin{pmatrix}1&2&3&4\\1&1&1&1\end{pmatrix}=\frac{1}{20}\begin{pmatrix}-6&-2&2&6\\20&10&0&-10\end{pmatrix}$$

(iii) The unknown vector **x** is given by $\mathbf{x}=\left[\left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\right]\mathbf{b}$ which is

$$\mathbf{x}=\begin{pmatrix}m\\c\end{pmatrix}=\frac{1}{20}\begin{pmatrix}-6&-2&2&6\\20&10&0&-10\end{pmatrix}\begin{pmatrix}1.7\\2.6\\4\\5\end{pmatrix}=\frac{1}{20}\begin{pmatrix}22.6\\10\end{pmatrix}=\begin{pmatrix}1.13\\0.50\end{pmatrix}$$

(iv) Substituting $m=1.13$ and $c=0.5$ into $y=mx+c$ gives $y=1.13x+0.5$. Plotting this line and the data points $(1,\ 1.7)$, $(2,\ 2.6)$, $(3,\ 4)$, $(4,\ 5)$ we have



57. Using the given table:

| Alphabet | A | B | C | D | … | W | X | Y | Z | |
|---|---|---|---|---|---|---|---|---|---|---|
| Position | 1 | 2 | 3 | 4 | … | 23 | 24 | 25 | 26 | 27 |
| Position $+3$ | 4 | 5 | 6 | 7 | … | 26 | 27 | 28 | 29 | 30 |

We have

| S | H | A | L | L | | W | E | | M | E | E | T |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 22 | 11 | 4 | 15 | 15 | 30 | 26 | 8 | 30 | 16 | 8 | 8 | 23 |

Writing out the 3 by 1 column vectors gives

$$\begin{pmatrix}22\\11\\4\end{pmatrix},\ \begin{pmatrix}15\\15\\30\end{pmatrix},\ \begin{pmatrix}26\\8\\30\end{pmatrix},\ \begin{pmatrix}16\\8\\8\end{pmatrix} \text{ and } \begin{pmatrix}23\\27\\27\end{pmatrix}$$

In order to complete the last vector we had to add two spaces. Multiplying each of these column vectors by the encoding matrix **A** gives

$$\begin{pmatrix}1&1&1\\0&1&1\\3&3&4\end{pmatrix}\begin{pmatrix}22&15&26&16&23\\11&15&8&8&27\\4&30&30&8&27\end{pmatrix}=\begin{pmatrix}37&60&64&32&77\\15&45&38&16&54\\115&210&222&104&258\end{pmatrix}=\mathbf{B}$$

The message is transmitted as

$$37,\ 15,\ 115,\ 60,\ 45,\ 210,\ 64,\ 38,\ 222,\ 32,\ 16,\ 104,\ 77,\ 54 \text{ and } 258$$

*How do we find the decoding matrix?*

The inverse matrix $\mathbf{A}^{-1}$ is the decoding matrix

$$\mathbf{A}^{-1}=\begin{pmatrix}1&1&1\\0&1&1\\3&3&4\end{pmatrix}^{-1}=\begin{pmatrix}1&-1&0\\3&1&-1\\-3&0&1\end{pmatrix}$$

In order to decode the message we have to find $\mathbf{A}^{-1}\mathbf{B}$:

$$\begin{pmatrix}1&-1&0\\3&1&-1\\-3&0&1\end{pmatrix}\begin{pmatrix}37&60&64&32&77\\15&45&38&16&54\\115&210&222&104&258\end{pmatrix}=\begin{pmatrix}22&15&26&16&23\\11&15&8&8&27\\4&30&30&8&27\end{pmatrix}$$

The columns of the last matrix gives us our initial message.

58. *Proof*.

Rearranging the given equation

$$\mathbf{AB}=-\mathbf{BA} \qquad (*)$$

Left multiplying this by **A** gives

$$\mathbf{A}^2\mathbf{B}=\mathbf{A}(-\mathbf{BA})=(-\mathbf{AB})\mathbf{A}=(\mathbf{BA})\mathbf{A}=\mathbf{BA}^2$$

We use this result $\mathbf{A}^2\mathbf{B} = \mathbf{B}\mathbf{A}^2$ to produce a proof of the given result. Right multiplying this $\mathbf{A}^2\mathbf{B} = \mathbf{B}\mathbf{A}^2$ by $\mathbf{B}^2$ gives

$$\begin{aligned}
\mathbf{A}^2\mathbf{B}^3 &= \mathbf{B}\mathbf{A}^2\mathbf{B}^2 \\
&= \mathbf{BA}(\mathbf{AB})\mathbf{B} \\
&= \mathbf{BA}(-\mathbf{BA})\mathbf{B} && [\text{By } (*)] \\
&= -\mathbf{B}(\mathbf{AB})\mathbf{AB} \\
&= \mathbf{B}(\mathbf{BA})(\mathbf{AB}) && [\text{By } (*)] \\
&= \mathbf{B}^2\mathbf{A}(\mathbf{AB}) \\
&= \mathbf{B}^2\mathbf{A}(-\mathbf{BA}) && [\text{By } (*)] \\
&= \mathbf{B}^2(-\mathbf{AB})\mathbf{A} = \mathbf{B}^2(\mathbf{BA})\mathbf{A} = \mathbf{B}^3\mathbf{A}^2
\end{aligned}$$

We have proven $\mathbf{A}^2\mathbf{B}^3 = \mathbf{B}^3\mathbf{A}^2$.

■

59. We use mathematical induction to prove this result.
*Proof*.

Clearly for $n = 1$ we have our required result $\mathbf{D} = \begin{pmatrix} e^{2x} & 0 \\ 0 & e^{x} \end{pmatrix}$.

Assume the result is true for $n = k$:

$$\mathbf{D}^k = \begin{pmatrix} e^{2kx} & 0 \\ 0 & e^{kx} \end{pmatrix} \qquad (\dagger)$$

We need to prove the case $n = k + 1$ which means we need to prove

$$\mathbf{D}^{k+1} = \begin{pmatrix} e^{2(k+1)x} & 0 \\ 0 & e^{(k+1)x} \end{pmatrix}$$

Consider the Left Hand Side $\mathbf{D}^{k+1}$:

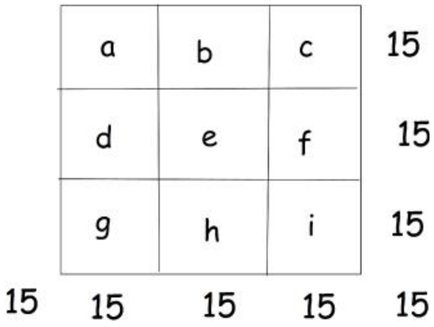
$$\mathbf{D}^{k+1} = \mathbf{D}^k\mathbf{D} = \begin{pmatrix} e^{2kx} & 0 \\ 0 & e^{kx} \end{pmatrix}\begin{pmatrix} e^{2x} & 0 \\ 0 & e^{x} \end{pmatrix}$$

$$= \begin{pmatrix} e^{2kx}e^{2x} & 0 \\ 0 & e^{kx}e^{x} \end{pmatrix} = \begin{pmatrix} e^{2kx+2x} & 0 \\ 0 & e^{kx+x} \end{pmatrix} = \begin{pmatrix} e^{2x(k+1)} & 0 \\ 0 & e^{x(k+1)} \end{pmatrix}$$

This is what was required, so by mathematical induction we have our result.

■

60. The magic square is given by




The linear system relating to this magic square is:

$$
\begin{aligned}
a + b + c &= 15\\
d + e + f &= 15\\
g + h + i &= 15\\
a + d + g &= 15\\
b + e + h &= 15\\
c + f + i &= 15\\
a + e + i &= 15\\
c + e + g &= 15
\end{aligned}
$$

Writing this matrix form gives

$$
\begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1\\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0\\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0\\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1\\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1\\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0
\end{pmatrix}
\begin{pmatrix} a\\ b\\ c\\ d\\ e\\ f\\ g\\ h\\ i \end{pmatrix}
=
\begin{pmatrix} 15\\ 15\\ 15\\ 15\\ 15\\ 15\\ 15\\ 15 \end{pmatrix}
$$

Entering the above as an augmented matrix and finding the reduced row echelon form of this gives:

$$
\begin{array}{c}
\\ R_1\\ R_2\\ R_3\\ R_4\\ R_5\\ R_6\\ R_7\\ R_8
\end{array}
\begin{array}{c}
\begin{array}{ccccccccc|c}
a & b & c & d & e & f & g & h & i & \\
\end{array}\\
\left(\begin{array}{ccccccccc|c}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 10\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 10\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & -5\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -2 & -10\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 5\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 2 & 20\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 15\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)
\end{array}
$$

Examining the 5th row, $R_5$, we have

$$e = 5$$

*Which ones are our free variables?*

By looking at the bottom non-zero row, $\mathrm{R}_7$, we can say the free variables are $h$ and $i$.

By expanding this row we have

$$g + h + i = 15 \Rightarrow g = 15 - h - i$$

Remember our unknowns need to be between 1 and 9. Let $h = 7,\ i = 2$ (you could chose other numbers such as $h = 9,\ i = 2$) then

$$g = 6$$

Substituting these values $h = 7,\ i = 2$ into $\mathrm{R}_6$ gives

$$f = 20 - h - 2i = 20 - 7 - 4 = 9$$

Similarly from $R_4$ we have

$$d = -10 + 4 + 7 = 1$$

By expanding the top two rows we have $a = 8$, $b = 3$ and $c = 4$.

Entering these values into the magic square gives

| | | | | |
|---|---|---|---|---|
| | 8 | 3 | 4 | 15 |
| | 1 | 5 | 9 | 15 |
| | 6 | 7 | 2 | 15 |
| 15 | 15 | 15 | 15 | 15 |