<u>Problems on Chapter 7</u>

1. (a) By finding the eigenvalues of matrix $\mathbf{A}=\begin{pmatrix}1 & 2\\ 0 & -1\end{pmatrix}$ show that $\mathbf{A}^{-1}=\mathbf{A}$.

   (b) Write $\mathbf{A}^{-1}$ in terms of the matrix **A** where $\mathbf{A}=\begin{pmatrix}1 & 2\\ 1 & -1\end{pmatrix}$.

2. Find the eigenvalues and eigenvectors of $\mathbf{A}=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$.

3. Find the eigenvalues and eigenvectors of $\mathbf{A}=\begin{pmatrix}1 & 2 & 3\\ 0 & 1 & 4\\ 0 & 0 & 1\end{pmatrix}$.

4. Let **A** be a square matrix. Describe the geometric effect of the following:
   (a) **A** has the eigenvector **x** corresponding to the eigenvalue 5.
   (b) **A** has the eigenvector **x** corresponding to the eigenvalue -2.

5. Let matrix **A** have an eigenvector $\mathbf{u}=\begin{pmatrix}2\\ 1\end{pmatrix}$ which belongs to the eigenvalue a. Draw graphs of $a\mathbf{u}$ for the following eigenvalues:
   (i) $a=1$ (ii) $a=-1$ (iii) $a=-2$ (iv) $a=0$

6. Let $\mathbf{A}=\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}$ which has an one eigenvalue equal to 5.74. Determine the other eigenvalue of this matrix.

7. Let $\mathbf{A}=\begin{pmatrix}a & b\\ 0 & c\end{pmatrix}$. Show that a is an eigenvalue with the eigenvector $\mathbf{x}=\begin{pmatrix}1\\ 0\end{pmatrix}$ of matrix **A**.

8. Decide whether $\mathbf{x}=\begin{pmatrix}0\\ 3/2\\ -1\end{pmatrix}$ is an eigenvector of $\mathbf{A}=\begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 1 & 0 & 1\end{pmatrix}$. If **x** is an eigenvector then find the corresponding eigenvalue.

9. Let the matrix **A** have the eigenvector **u** belonging to the eigenvalue a. Show that $c\mathbf{u}$ is also an eigenvector of **A** with the same eigenvalue for any non-zero scalar *c*.

10. Let matrix **A** act on a vector **u** so that it produces the following transform:

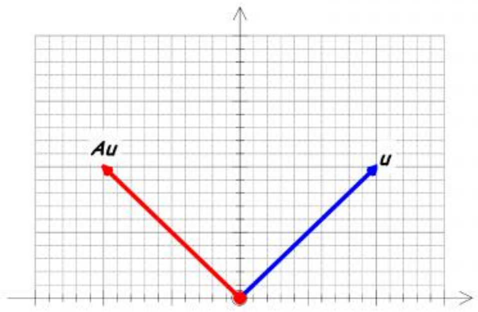

Figure 1
Explain why **u** is not an eigenvector of matrix **A** belonging to a real eigenvalue.

11. (i) Determine the eigenvalues of $\mathbf{A}=\begin{pmatrix}2 & 1\\ 2 & 3\end{pmatrix}$.

(ii) Find an invertible matrix **P** such that $\mathbf{P}^{-1}\mathbf{AP}$ is a diagonal matrix.
(iii) Find the diagonal matrix **D** such that $\mathbf{D}^2=\mathbf{P}^{-1}\mathbf{AP}$.
(iv) Determine a matrix **B** such that $\mathbf{B}=\mathbf{PDP}^{-1}$ and find $\mathbf{B}^2$. *What do you notice about your result?*

12. Let $\mathbf{A}=\begin{pmatrix}5 & -4\\ -4 & 5\end{pmatrix}$. By using the power formula for matrices $\mathbf{A}^n=\mathbf{PD}^n\mathbf{P}^{-1}$ where **P** is the eigenvector matrix find $\sqrt{\mathbf{A}}$.

13. Given that $\mathbf{A}^2=\mathbf{I}$ where **I** is the 2 by 2 identity matrix, find $\sqrt{\mathbf{A}}\neq\mathbf{I}$.

14. Explain why we cannot diagonalize the matrix $\mathbf{A}=\begin{pmatrix}0 & 4\\ -1 & 4\end{pmatrix}$.

15. Let $\mathbf{A}=\begin{pmatrix}4 & 3\\ 1 & 2\end{pmatrix}$. Determine an invertible matrix **P** and a diagonal matrix **D** such that $\mathbf{P}^{-1}\mathbf{AP}=\mathbf{D}$. Also find $\mathbf{A}^5$.

16. Let $\mathbf{A}=\begin{pmatrix}1 & 5 & 7\\ 0 & 2 & 5\\ 0 & 0 & 3\end{pmatrix}$. Show that $\mathbf{A}^3-6\mathbf{A}^2+11\mathbf{A}-6\mathbf{I}=\mathbf{O}$.

17. Prove that if matrices **A** and **B** are similar then $\det(\mathbf{A})=\det(\mathbf{B})$.

18. Prove that if matrix **A** has null space equal to $\{\mathbf{O}\}$ then **A** does not have 0 as an eigenvalue.

19. Construct a 3 by 3 matrix which has the characteristic polynomial given by

$$\left(a^2 - 4\right)\left(a + 3\right)$$

20. Let **A** be a 3 by 3 matrix. Given that the eigenvalues of matrix **A** are 1, 2 and 3, find the inverse matrix $\mathbf{A}^{-1}$ as an expression of **A**.

21. In *discrete dynamical systems*, we are interested in applying matrix **A** to a vector **x** and then using this result, $\mathbf{Ax}$, as the input to the matrix **A**. This means we are interested in:

$$\mathbf{Ax},\ \mathbf{A}(\mathbf{Ax}),\ \mathbf{A}\left(\mathbf{A}^2\mathbf{x}\right),\ \cdots,\ \mathbf{A}\left(\mathbf{A}^n\mathbf{x}\right)$$

Let $\mathbf{A} = \begin{pmatrix} 1 & 0.5 \\ 0 & 0.5 \end{pmatrix}$ and $\mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Determine $\lim_{n\to\infty}\left(\mathbf{A}^n\mathbf{x}\right)$.

[You may assume that if $|x| < 1$ then $\lim_{n\to\infty}\left(x^n\right) = 0$.]

22. Female population growth can be described by a Leslie matrix **L** applied to an initial vector population $\mathbf{x}_0$. The population after *n* years denoted $\mathbf{x}_n$ is given by

$$\mathbf{x}_n = \mathbf{L}^n\mathbf{x}_0$$

Let $\mathbf{L} = \begin{pmatrix} 0 & 4 & 1 \\ 0.2 & 0 & 0 \\ 0 & 0.1 & 0 \end{pmatrix}$ and $\mathbf{x}_0 = \begin{pmatrix} 100 \\ 200 \\ 300 \end{pmatrix}$. The eigenvalues of matrix **L** are

$$a_1 = 0.91,\ a_2 = -0.88 \text{ and } a_3 = -0.03$$

What can you say about the long term population, that is find the vector $\mathbf{x}_n$ as $n \to \infty$?

[Hint: You may assume that $\lim_{n\to\infty}\left(x^n\right) = 0$ provided $|x| < 1$.]

23. Let **A** be a 2 by 2 matrix with eigenvalues a_1 and a_2 with the corresponding eigenvectors **u** and **v**. If $\mathbf{w} = c\mathbf{u} + k\mathbf{v}$ where *k* and *c* are scalars, prove that

$$\mathbf{A}^n\mathbf{w} = c a_1^n\mathbf{u} + k a_2^n\mathbf{v} \quad (n \text{ is a natural number})$$

By using this formula determine $\mathbf{A}^{10}\mathbf{w}$ where $\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

24. Populations of predator and prey is given by the discrete dynamical system

$$\mathbf{u}_n = \mathbf{A}^n\mathbf{u}_0$$

where $\mathbf{u}_0$ is the initial population vector, $\mathbf{u}_n$ is the population vector at stage n and **A** is a square matrix. By using the formula of the previous question find $\mathbf{u}_n$ for

$\mathbf{A} = \begin{pmatrix} 6 & -1 \\ 4 & 1 \end{pmatrix}$ and

(a) $\mathbf{u}_0 = \begin{pmatrix} 100 \\ 100 \end{pmatrix}$ (b) $\mathbf{u}_0 = \begin{pmatrix} 100 \\ 400 \end{pmatrix}$ (c) $\mathbf{u}_0 = \begin{pmatrix} 100 \\ 250 \end{pmatrix}$

25. Let $\mathbf{A}^n$ be a 2 by 2 matrix with eigenvectors **u** and **v**. Show that the matrix **A** may have different eigenvectors from $\mathbf{A}^n$ where *n* is a natural number ≥ 2.

26. The matrix $\mathbf{M} = \begin{pmatrix} \cos(\text{¯}) & \sin(\text{¯}) & 0 \\ -\sin(\text{¯}) & \cos(\text{¯}) & 0 \\ 0 & 0 & 1 \end{pmatrix}$ represents a rotation of angle ¯ about the *z* axis, where $0 < \text{¯} < 180°$. Find the *real* eigenvalues and eigenvectors of **M** and comment on your answers.

27. (Markov chains). The urban and rural population of the UK is 56 and 6 million respectively. We can represent this as a column vector $\mathbf{v}_0 = \begin{pmatrix} 56 \\ 6 \end{pmatrix}$.

Suppose during a certain period, the probability of a person moving from urban (U) to rural (R) is 0.08 and from rural to urban is 0.05. These probabilities can be represented by the following transition matrix **T**:

$$\begin{matrix} & \text{From} & \text{U} & \text{R} & & \\ & \mathbf{T} = & \Big(0.92 & 0.05 \Big) & \text{U} & \\ & & \Big(0.08 & 0.95 \Big) & \text{R} & \text{To} \end{matrix}$$

In *n* years the population distribution $\mathbf{v}_n$ is given by $\mathbf{v}_n = \mathbf{T}^n \mathbf{v}_0$. Find a formula for $\mathbf{v}_n$ as $n \to \infty$. This $\mathbf{v}_n$ as $n \to \infty$ is the long term population distribution.

[You may find the following result useful: If $|x| < 1$ then $\lim_{n \to \infty} (x^n) = 0$.]

28. Let $\mathbf{A} = \begin{pmatrix} 2 & 5 \\ 0 & 3 \end{pmatrix}$. Determine the trace (the sum of the leading entries) of $\mathbf{A}^n$ where *n* is a natural number.

29. Find the eigenvalues and corresponding eigenvectors of the following matrix:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 4 & -2 \\ 2 & -1 & 0 & 1 \\ 2 & -1 & -1 & 2 \end{pmatrix}$$

30. Let **A** be any matrix. The norm of matrix **A** denoted by $\|\mathbf{A}\|$ is defined as

$$\|\mathbf{A}\| = \sqrt{\max\left\{\|\mathbf{Ax}\|^2 \mid \|\mathbf{x}\| = 1\right\}}$$

This notation can be off putting but we can write this as:

$$\|\mathbf{A}\|^2 = \max\left\{\|\mathbf{Ax}\|^2 \mid \|\mathbf{x}\| = 1\right\}$$

This means that norm of a matrix squared is the maximum value of $\|\mathbf{Ax}\|^2$ such that **x** is a unit vector.

The maximum value of $\|\mathbf{Ax}\|^2$ is the absolute value of the largest eigenvalue of the symmetric matrix $\mathbf{A}^T\mathbf{A}$.

(i) Show that $\|\mathbf{Ax}\|^2 = \mathbf{x}^T\left(\mathbf{A}^T\mathbf{A}\right)\mathbf{x}$.

(ii) Show that $\mathbf{A}^T\mathbf{A}$ is a symmetric matrix.

(iii) Determine $\|\mathbf{A}\|$ for $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$. Use Matlab for this part.

(iv) By using Matlab enter the matrix **A** of part (iii) and then enter command norm(A). What do you notice about your result?

31. Is $\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ an eigenvector of $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{pmatrix}$?

32. Let $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 1 \end{pmatrix}$.

(i) Determine the *reduced* SVD of **A**.
(ii) Determine the *full* SVD of **A**.

33. Decide whether the following matrices are similar:

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 2 & 3 & 1 \end{pmatrix}$$

34. Let $\mathbf{A}=\begin{pmatrix}5 & 6 & 2\\ 0 & -1 & -8\\ 1 & 0 & -2\end{pmatrix}$ and two of the eigenvalues of this matrix are $a_1=-4$ and $a_2=3$.

(i) Determine the third eigenvalue.
(ii) Is the matrix **A** invertible?
(iii) Find the determinant of matrix **A**.

35. Consider the orthogonal matrix $\mathbf{A}=\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & 1 & -1 & -1\\ 1 & -1 & 1 & -1\\ 1 & -1 & -1 & 1\end{pmatrix}$. Find an orthogonal matrix **Q** such that $\mathbf{Q}^{-1}\mathbf{A}\mathbf{Q}$ is diagonal.

36. Let **A** be a matrix with eigenvalue a with the corresponding eigenvector **v**. Show that the eigenvalue of matrix $k\mathbf{A}$ where $k \neq 0$ is ka with the same eigenvector. Find an orthogonal matrix of $\frac{1}{2}\mathbf{A}$ where **A** is the matrix given in the previous question.

37. Let **A** be a 5 by 5 matrix with eigenvalues 1, 2, 3 , 4 and 5. Find the eigenvalues of
(a) $3\mathbf{A}$ (b) $\mathbf{A}^T$ (c) $\mathbf{A}^{-1}$

38. Let **A** be a 2 by 2 orthogonal matrix with an eigenvalue a . Show that the other eigenvalue of **A** is $\pm\frac{1}{a}$.

39. If a matrix **A** has distinct eigenvalues then is matrix **A** invertible (non-singular)?

40. Let matrix **A** have eigenvalue a . Determine the eigenvalue of $\mathbf{A}+n\mathbf{I}$ where $n \in \mathbb{N}$.

41. A square matrix **A** is called a stochastic matrix if all entries in each of the column vectors of **A** are real non-negative numbers and the column sum is 1.
An example is $\mathbf{A}=\begin{pmatrix}1/2 & 1/2\\ 1/2 & 1/2\end{pmatrix}$. Find a formula for $\mathbf{A}^m$.

42. Let **A** be the 2 by 2 stochastic matrix and **x** be a 2 by 1 stochastic column vector. Show that $\mathbf{Ax}$ is a stochastic vector which means that the sum of the entries in this vector equal 1.

43. Let **A** be a 2 by 2 stochastic matrix given by

$$\mathbf{A}=\begin{pmatrix}1-a & b\\ a & 1-b\end{pmatrix} \text{ where } 0 \le a,\ b \le 1$$

Show that this matrix **A** has the eigenvectors $\mathbf{u}=\begin{pmatrix}1\\-1\end{pmatrix}$ and $\mathbf{v}=\begin{pmatrix}b\\a\end{pmatrix}$ and find the corresponding eigenvalues.

44. A *discrete dynamical* system is defined by

$$\mathbf{v}_k=\mathbf{A}^k\mathbf{v}_0 \qquad (k=1,\ 2,\ 3,\ \cdots)$$

where **A** is a *n* by *n* matrix and $\mathbf{v}_k$, $\mathbf{v}_0$ are *n* by 1 column vectors. The *k*th vector $\mathbf{v}_k$ is found from the initial vector $\mathbf{v}_0$ by applying the matrix $\mathbf{A}^k$ to it.

Find $\mathbf{v}_{10}$ given that $\mathbf{A}=\begin{pmatrix}1&0&2\\0&1&0\\2&0&1\end{pmatrix}$ and $\mathbf{v}_0=\begin{pmatrix}1\\1\\1\end{pmatrix}$.

45. In a course on differential equations you normally have to solve the system:

$\mathbf{x}'=\mathbf{A}\mathbf{x}$ where **A** is a square matrix and $\mathbf{x}'$, $\mathbf{x}$ are column vectors

The general solution of this system is given by

$$\mathbf{x}=c_1\mathbf{u}_1e^{a_1 t}+c_2\mathbf{u}_2e^{a_2 t}+\cdots+c_n\mathbf{u}_ne^{a_n t}$$

where **u** is the eigenvector belonging to the eigenvalue a of the matrix **A** and the eigenvectors are linearly independent. (The *c*'s are scalars.)

Find the general solution of

(a) $\mathbf{x}'=\begin{pmatrix}3&0&4\\0&5&0\\4&0&3\end{pmatrix}\mathbf{x}$ (b) $\mathbf{x}'=\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}\mathbf{x}$

46. Let **A** be a *n* by *n* diagonalizable matrix such that

$$\mathbf{A}=\mathbf{PDP}^{-1}$$

where **P** is the eigenvector matrix and **D** is the eigenvalue (diagonal) matrix given by

$$\mathbf{D}=\begin{pmatrix}a_1&&0\\&\ddots&\\0&&a_n\end{pmatrix}$$ $a_1,\ a_2,\ \cdots,\ a_n$ are the eigenvalues of **A**

The *matrix exponential* $e^{t\mathbf{A}}$ is defined as

$$e^{t\mathbf{A}}=\mathbf{P}\begin{pmatrix}e^{a_1 t}&&0\\&\ddots&\\0&&e^{a_n t}\end{pmatrix}\mathbf{P}^{-1}$$

Determine the $e^{t\mathbf{A}}$ for $\mathbf{A}=\begin{pmatrix}1&1\\4&-2\end{pmatrix}$.

47. If we consider a *system of differential equations* defined above with initial conditions

$$\mathbf{x}'=\mathbf{A}\mathbf{x} \quad \text{and} \quad \mathbf{x}(0)=\mathbf{x}_0$$

where **A** is a square matrix and $\mathbf{x}'$, $\mathbf{x}$ are column vectors, $\mathbf{x}(0)=\mathbf{x}_0$ means the vector **x** at $t=0$ is $\mathbf{x}_0$.

The solution of this system is given by

$$\mathbf{x}(t)=e^{t\mathbf{A}}\mathbf{x}_0$$

Solve the system of differential equations for the matrix **A** given in the previous question and $\mathbf{x}_0=\begin{pmatrix}1\\2\end{pmatrix}$.

48. Prove that matrices **A** and **B** are similar matrices $\Leftrightarrow$ **A** and **B** are diagonalizable with the same diagonal matrix **D**.

49. Test whether matrices $\mathbf{A}=\begin{pmatrix}1&1\\1&1\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}-1&-3\\1&3\end{pmatrix}$ are similar.

50. Show that matrices $\mathbf{A}=\begin{pmatrix}2&0&0\\0&2&0\\0&0&2\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}2&1&3\\0&2&4\\0&0&2\end{pmatrix}$ are not similar.

    [Hint: Use the result of question 48.]

51. Find the eigenvalues of the following matrices:

    (a) $\mathbf{A}=\begin{pmatrix}0&1\\2&0\end{pmatrix}$, (b) $\mathbf{B}=\begin{pmatrix}0&0&0&1\\0&0&2&0\\0&3&0&0\\4&0&0&0\end{pmatrix}$ and (c) $\mathbf{C}=\begin{pmatrix}0&0&0&0&0&1\\0&0&0&0&2&0\\0&0&0&3&0&0\\0&0&4&0&0&0\\0&5&0&0&0&0\\6&0&0&0&0&0\end{pmatrix}$

    Predict a formula for evaluating the eigenvalues of a $2n$ by $2n$ matrix with entries $\sim_1,\ \sim_2,\ \cdots,\ \sim_{2n}$ on the secondary diagonal and zeros elsewhere.

Brief Solutions to Problems 7

1. (b) $\frac{1}{3}\mathbf{A}$

2. $a_{1,\ 2} = 0$, $\mathbf{u} = s\begin{pmatrix}1\\0\end{pmatrix}$, $\left\{\begin{pmatrix}0\\1\end{pmatrix}\right\}$

3. $a_{1,2,3} = 1$, $\mathbf{u} = s\begin{pmatrix}1\\0\\0\end{pmatrix}$

6. -0.74

8. $a = 1$

11. (i) $a_1 = 1,\ a_2 = 4$ (ii) $\begin{pmatrix}-1 & 1\\ 1 & 2\end{pmatrix}$ (iii) $\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}$ (iv) $\frac{1}{3}\begin{pmatrix}4 & 1\\ 2 & 5\end{pmatrix}$, $\mathbf{B}^2 = \mathbf{A}$

12. $\begin{pmatrix}2 & -1\\ -1 & 2\end{pmatrix}$

13. $\mathbf{A} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$

15. $\mathbf{P} = \begin{pmatrix}3 & 1\\ 1 & -1\end{pmatrix}$, $\mathbf{D} = \begin{pmatrix}5 & 0\\ 0 & 1\end{pmatrix}$, $\mathbf{A}^5 = \begin{pmatrix}2344 & 2343\\ 781 & 782\end{pmatrix}$

20. $\frac{1}{6}\left(\mathbf{A}^2 - 6\mathbf{A} + 11\mathbf{I}\right)$

21. $\begin{pmatrix}2\\0\end{pmatrix}$

22. $\mathbf{x}_n \to \mathbf{O}$

23. $\begin{pmatrix}3070\\1024\end{pmatrix}$

24. (a) $5^n\mathbf{u}_0$ (b) $2^n\mathbf{u}_0$ (c) $50\begin{pmatrix}5^n + 2^n\\ 5^n + 4\left(2^n\right)\end{pmatrix}$

25. $\mathbf{A} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ has $\mathbf{u} = \begin{pmatrix}1\\-1\end{pmatrix}$, $\mathbf{v} = \begin{pmatrix}1\\1\end{pmatrix}$ but $\mathbf{A}^2 = \mathbf{I}$ has $\mathbf{u}' = \begin{pmatrix}1\\0\end{pmatrix}$, $\mathbf{v}' = \begin{pmatrix}0\\1\end{pmatrix}$

26. $a_1 = 1$ is $\mathbf{u} = \begin{pmatrix}0\\0\\1\end{pmatrix}$

27. $\mathbf{v}_n = \begin{pmatrix}23.85\\38.15\end{pmatrix}$ as $n \to \infty$.

28. $2^n + 3^n$

29. $a_{1,\ 2,\ 3,\ 4} = 1,\ \mathbf{u} = \begin{pmatrix} 1 \\ 3 \\ 1 \\ 2 \end{pmatrix},\ \mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix}$

30. (iii) 9.508

31. No.

32. (i) $\frac{1}{\sqrt{5}} \begin{pmatrix} 2/\sqrt{30} & -1/\sqrt{5} \\ 1/\sqrt{30} & 2/\sqrt{5} \\ 5/\sqrt{30} & 0 \end{pmatrix} \begin{pmatrix} \sqrt{6} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$

(ii) $\frac{1}{\sqrt{5}} \begin{pmatrix} 2/\sqrt{30} & -1/\sqrt{5} & 2/\sqrt{6} \\ 1/\sqrt{30} & 2/\sqrt{5} & 1/\sqrt{6} \\ 5/\sqrt{30} & 0 & -1/\sqrt{6} \end{pmatrix} \begin{pmatrix} \sqrt{6} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$

33. No.

34. (i) $a_3 = 3$ (ii) Yes (iii) -36

35. $\mathbf{Q} = \begin{pmatrix} \hat{\mathbf{u}} & \hat{\mathbf{v}} & \hat{\mathbf{w}} & \hat{\mathbf{x}} \end{pmatrix}$, $\hat{\mathbf{u}} = \frac{1}{\sqrt{12}} \begin{pmatrix} 3 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $\hat{\mathbf{v}} = \frac{1}{\sqrt{6}} \begin{pmatrix} 0 \\ -2 \\ 1 \\ 1 \end{pmatrix}$, $\hat{\mathbf{w}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}$ and $\hat{\mathbf{x}} = \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ -1 \\ -1 \end{pmatrix}$

36. Same **Q** as solution 35.

37. (a) 3, 6, 9 , 12 and 15 (b) 1, 2, 3 , 4 and 5 (c) $1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4}$ and $\frac{1}{5}$

39. No.

40. $a + n$

41. $\mathbf{A}^m = \mathbf{A}$

44. $\begin{pmatrix} 59049 \\ 1 \\ 59049 \end{pmatrix}$

45. (a) $c_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{5t} + c_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{7t}$ (b) $c_1 \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{3t}$

46. $\frac{1}{5} \begin{pmatrix} 4e^{2t} + e^{-3t} & e^{2t} - e^{-3t} \\ 4e^{2t} - 4e^{-3t} & e^{2t} + 4e^{-3t} \end{pmatrix}$

47. $\frac{1}{5} \begin{pmatrix} 6e^{2t} - e^{-3t} \\ 6e^{2t} + 4e^{-3t} \end{pmatrix}$

49. Yes.

51. (a) $\pm\sqrt{2}$ (b) $\pm\sqrt{6},\ \pm 2$ (c) $\pm\sqrt{6}, \pm\sqrt{10},\ \pm\sqrt{12}$

$a_{1,\ 2} = \pm\sqrt{\sim_1 \times \sim_{2n}},\ a_{3,\ 4} = \pm\sqrt{\sim_2 \times \sim_{2n-1}},\ a_{5,\ 6} = \pm\sqrt{\sim_3 \times \sim_{2n-2}},\ \ldots,\ a_{2n-1,\ 2n} = \sqrt{\sim_n \sim_{n+1}}$