Problems on Chapter 6: Determinants

1. Show that $\det(2\mathbf{A}) \neq \det(\mathbf{A}) + \det(\mathbf{A})$.

2. By using determinant, find the values of *k* so that the following vectors do not form a basis for $\mathbb{R}^3$:

$$\begin{pmatrix} k \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ k \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ k \end{pmatrix}$$

3. Let **A** and **B** be 6 by 6 matrices and $\det(\mathbf{A}) = -2$, $\det(\mathbf{B}) = 3$. Find

(a) $\det(\mathbf{A}^3)$ (b) $\det(5\mathbf{B})$ (c) $\det\left((\mathbf{AB})^5\right)$ (d) $\det(\mathbf{A}^{-1}\mathbf{B}^{-1})$

4. Find $\det(\mathbf{A})$ where $\mathbf{A} = \begin{pmatrix} 1001 & 2001 & 3001 \\ 1002 & 2002 & 3002 \\ 1003 & 2003 & 3003 \end{pmatrix}$.

5. Find the determinant of the following matrix without using a calculator:

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 6 & 7 & 1 & 8 \\ 2 & 1 & 3 & 4 \end{pmatrix}$$

6. Let $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$ be our standard basis for $\mathbb{R}^n$. Show that

$$\det\begin{pmatrix} x_1\mathbf{e}_1 & x_2\mathbf{e}_2 & \cdots & x_n\mathbf{e}_n \end{pmatrix} = x_1 x_2 \cdots x_n$$

7. Show that the transformation $T: M_{nn} \to \mathbb{R}$ given by

$T(\mathbf{A}) = \det(\mathbf{A})$ where **A** is a *n* by *n* matrix

is not linear.

8. Prove the following result:

$\det\begin{pmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_n \end{pmatrix} \neq 0 \quad \Leftrightarrow \quad \{\mathbf{c}_1, \mathbf{c}_2, \cdots, \mathbf{c}_n\}$ form a basis for $\mathbb{R}^n$

9. Determine the following shaded area:

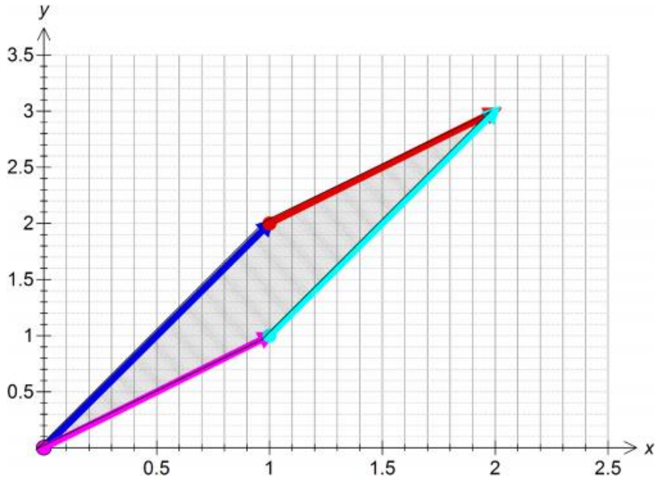

Figure 1

10. The equation of a line through two points $P=(x_1, \quad y_1)$ and $Q=(x_2, \quad y_2)$ is given by

$$\det\begin{pmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{pmatrix}=0$$

Find the equation of the line through the points $(5, \quad 6)$ and $(6, \quad 5)$ and sketch these points and the line on a graph.

11. Find $\det(\mathbf{A})$ where $\mathbf{A}=\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

12. Find $\det(\mathbf{A})$ where $\mathbf{A}=\begin{pmatrix} a & b & 0 & 0 & 0 \\ 0 & a & b & 0 & 0 \\ 0 & 0 & a & b & 0 \\ 0 & 0 & 0 & a & b \\ b & 0 & 0 & 0 & a \end{pmatrix}$.

13. Let $\mathbf{A}_n$ be the *n* by *n* matrix with 1's along the secondary diagonal and zeros everywhere else in the matrix. For example $\mathbf{A}_3=\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

(i) Find the determinants of $\mathbf{A}_2, \ \mathbf{A}_3, \ \mathbf{A}_4, \ \mathbf{A}_5, \mathbf{A}_6$.

(ii) Predict a formula for $\det(\mathbf{A}_n)$.

14. Let $\mathbf{A}=\begin{pmatrix}13 & 4\\ 1 & 1\end{pmatrix}$. Determine the matrix **X** given by

$$\mathbf{X}=\frac{1}{\sqrt{trace(\mathbf{A})+2\sqrt{\det(\mathbf{A})}}}\left[\mathbf{A}+\left(\sqrt{\det(\mathbf{A})}\times\mathbf{I}\right)\right]$$

Where the trace of the matrix is sum of all the leading terms and **I** is the identity matrix. Also find $\mathbf{X}^2$. What do you notice about your result?

15. Let $\mathbf{U}=\begin{pmatrix}1&1&1&1&1\\0&1&2&3&4\\0&0&1&3&6\\0&0&0&1&4\\0&0&0&0&1\end{pmatrix}$ and $\mathbf{L}=\begin{pmatrix}1&0&0&0&0\\1&1&0&0&0\\1&2&1&0&0\\1&3&3&1&0\\1&4&6&4&1\end{pmatrix}$. The Pascal matrix $\mathbf{P}_5$ is a matrix whose entries are the binomial coefficients of an expansion to the index 5.

(i) Show that $\mathbf{P}_5=\mathbf{LU}=\begin{pmatrix}1&1&1&1&1\\1&2&3&4&5\\1&3&6&10&15\\1&4&10&20&35\\1&5&15&35&70\end{pmatrix}$.

(ii) Find the determinants of $\mathbf{L}$, $\mathbf{U}$ and $\mathbf{P}_5$.

16. Find a non-zero 3 by 3 matrix whose determinant is zero.

17. Either proof or provide a counter example of the following:
If **A** is a non-zero triangular matrix then **A** is invertible.

18. Show that for same size square matrices **A** and **B**:

(i) $\det(\mathbf{AB}-\mathbf{BA})\neq 0$

(ii) $\det(\mathbf{AB})-\det(\mathbf{BA})=0$

19. Is any set of square matrices whose determinant is zero a vector space with respect to the usual matrix addition and scalar multiplication?

20. Prove that a triangular matrix with non-zero entries on the leading diagonal is invertible.

21. Show that a triangular matrix may not be invertible.

22. Show that the Wronskian $W\left(e^{2x}\cos(x),\quad e^{2x}\sin(x)\right)\neq 0$.

23. Find any errors in the following derivation. You must say what the error is and why. Let **A** and **B** be square matrices. Then

$$\begin{aligned}\det\left((\mathbf{A}+\mathbf{B})^2\right) &= \left[\det(\mathbf{A}+\mathbf{B})\right]^2 \\ &= \left[\det(\mathbf{A})+\det(\mathbf{B})\right]^2\end{aligned}$$

24. Let $\mathbf{A}=\begin{pmatrix} x & 1 & 1 & 1 \\ 1 & x & 1 & 1 \\ 1 & 1 & x & 1 \\ 1 & 1 & 1 & x \end{pmatrix}$. Find

(i) determinant of the matrix **A**.
(ii) the value of *x* for which the matrix is non-invertible.

25. An anti-symmetric matrix **A** is a square matrix such that $\mathbf{A}^T=-\mathbf{A}$.
Let **A** be an *n* by n anti-symmetric matrix.
(i) Prove that if *n* is odd then $\det(\mathbf{A})=0$.
(ii) Find a formula for $\det\left(\mathbf{A}^m\mathbf{A}^T\right)$.

Brief Solutions to Problems of Chapter 6

2. $k = -2,\ 1$
3. (a) -8 (b) $46\,875$ (c) -7776 (d) $-\frac{1}{6}$
4. 0
5. 40
9. 1
10. $y = 11 - x$
11. -1
12. $a^5 + b^5$
13. (i) $\det(\mathbf{A}_2) = \det(\mathbf{A}_3) = -1,\ \det(\mathbf{A}_4)\det(\mathbf{A}_5) = 1,\ \det(\mathbf{A}_6) = -1$

    (ii) $\det(\mathbf{A}_n) = (-1)^{\lfloor n/2 \rfloor}$
14. $\mathbf{X} = \frac{1}{\sqrt{20}}\begin{pmatrix} 16 & 4 \\ 1 & 4 \end{pmatrix},\ \mathbf{X}^2 = \mathbf{A}$
15. (ii) 1 in each case
17. The result is false.
19. No
23. Error in the last line.
24. (i) $\det(\mathbf{A}) = (x-1)^3(3+x)$ (ii) $1,\ -3$
25. (ii) $\begin{cases} \left[\det(\mathbf{A})\right]^{m+1} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$