<u>Problems on Chapter 4</u>

1. Transform the basis vectors $\left\{\mathbf{u}=\begin{pmatrix}1\\0\end{pmatrix},\ \mathbf{v}=\begin{pmatrix}0.999\\0.001\end{pmatrix}\right\}$ to an orthonormal basis.

   Write the vector $\mathbf{x}=\begin{pmatrix}1\\1\end{pmatrix}$ in terms of the above given basis.

2. If $\mathbf{A}=\mathbf{QB}$ where **Q** is an orthogonal matrix then show that

$$\mathbf{A}^T\mathbf{A}=\mathbf{B}^T\mathbf{B}$$

3. There are a number of different ways we can chose the norm of a matrix because M_{mn} is a vector space so a matrix norm is no different from a vector norm.
   One way is Frobenius norm, $\|\mathbf{A}\|_F$, of a matrix **A** of size *m* by *n* which is defined by

$$\|\mathbf{A}\|_F=\sqrt{\sum_{j=1}^{n}\sum_{i=1}^{m}\left(a_{ij}\right)^2}$$

   Determine $\|\mathbf{A}\|_F$ for (a) $\mathbf{A}=\begin{pmatrix}1&2\\3&4\end{pmatrix}$ (b) $\mathbf{A}=\begin{pmatrix}1&2&3&4&5\\6&7&8&9&10\end{pmatrix}$

4. Show that $\mathbf{A}=\begin{pmatrix}\cos(\text{¯})&-\sin(\text{¯})\\\sin(\text{¯})&\cos(\text{¯})\end{pmatrix}$ is an orthogonal matrix. Find $\mathbf{A}^{-1}$.

5. Show that $\mathbf{A}=\begin{pmatrix}\cos(\text{¯})&0&\sin(\text{¯})\\0&1&0\\-\sin(\text{¯})&0&\cos(\text{¯})\end{pmatrix}$ is orthogonal matrix and determine $\mathbf{A}^{-1}$.

   Determine $\mathbf{Ax}$ where $\mathbf{x}=\begin{pmatrix}1&1&1\end{pmatrix}^T$ and sketch **x** and **Ax** for
   (a) $\text{¯}=90°$ (b) $\text{¯}=180°$ (c) $\text{¯}=270°$
   Describe what the matrix **A** does to the vector **x** for each of these ¯ values.

6. For the matrix **A** given in question 5 determine $\mathbf{Ax}$ for the standard basis (axes) vectors $\{\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3\}$ for $\text{¯}=45°$. Describe what the matrix **A** does to these basis vectors. (Try making a sketch by using software.)

7. Let P_1 be the vector space of linear polynomials. The standard basis for P_1 is $\{1,\ x\}$.
   By using the inner product $\langle f,\ g\rangle=\int_{-1}^{1} f(x)g(x)\,\mathrm{d}x$ convert this standard basis to an orthonormal basis for P_1.

8. Show that $\mathbf{A}=\frac{1}{2}\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & 1 & -1 & -1\\ 1 & -1 & 1 & -1\\ 1 & -1 & -1 & 1\end{pmatrix}$ is an orthogonal matrix and find $\mathbf{A}^{-1}$.

9. Let **u** and **v** be vectors in a vector space. Find the first *error*, if any, in the following derivation:

$$\begin{aligned}\|\mathbf{u}+\mathbf{v}-(\mathbf{u}-\mathbf{v})\| &= \langle 2\mathbf{v},\ 2\mathbf{v}\rangle \\ &= 4\|\mathbf{v}\|^2\end{aligned}$$

10. Let $\mathbf{u}=\begin{pmatrix}a\\ b\end{pmatrix}$ be a non-zero vector in $\mathbb{R}^2$.

(i) Find an orthonormal basis of $\mathbb{R}^2$ such that normalized **u** is one of these basis vectors.

(ii) Give an example of a non-standard orthonormal basis for $\mathbb{R}^2$.

11. Let $\mathbf{u}=\begin{pmatrix}1\\ 0\\ 0\end{pmatrix}$, $\mathbf{v}=\begin{pmatrix}2\\ 0\\ 3\end{pmatrix}$ and $\mathbf{w}=\begin{pmatrix}4\\ 5\\ 6\end{pmatrix}$ be a basis for $\mathbb{R}^3$. Form an orthonormal basis for $\mathbb{R}^3$ from these vectors $\{\mathbf{u},\ \mathbf{v},\ \mathbf{w}\}$.

Factorise the matrix $\mathbf{A}=(\mathbf{u}\quad \mathbf{v}\quad \mathbf{w})$ into **QR** where **Q** is an orthogonal matrix and **R** is an upper triangular matrix.

12. (i) Transform the following vectors which span a subspace of $\mathbb{R}^4$ to an orthonormal basis for this subspace:

$$\begin{pmatrix}1\\ 2\\ 3\\ 4\end{pmatrix},\ \begin{pmatrix}1\\ 2\\ 0\\ 3\end{pmatrix} \text{ and } \begin{pmatrix}2\\ 3\\ 5\\ 8\end{pmatrix}$$

(ii) Find the **QR** factorization of the matrix $\mathbf{A}=\begin{pmatrix}1 & 1 & 2\\ 2 & 2 & 3\\ 3 & 0 & 5\\ 4 & 3 & 8\end{pmatrix}$.

13. Let $\mathbf{u}=\begin{pmatrix}1/2\\1/2\\1/2\end{pmatrix}$, $\mathbf{v}=\begin{pmatrix}1/3\\2/3\\1\end{pmatrix}$ and $\mathbf{w}=\begin{pmatrix}1/4\\1/4\\1/2\end{pmatrix}$ be a basis for $\mathbb{R}^3$. Form an orthonormal basis for $\mathbb{R}^3$ from these vectors $\{\mathbf{u},\ \mathbf{v},\ \mathbf{w}\}$.

14. In *Special Relativity* we use 4 coordinates, $\mathbb{R}^4$; 3 space coordinates $x,\ y,\ z$ and one time coordinate *t*. An important function in this geometry is defined as

$$\left\langle \begin{pmatrix}x_1 & y_1 & z_1 & t_1\end{pmatrix}^T,\ \begin{pmatrix}x_2 & y_2 & z_2 & t_2\end{pmatrix}^T \right\rangle = -x_1x_2 - y_1y_2 - z_1z_2 + t_1t_2 \qquad (*)$$

This is *not* quite an inner product because it fails the following axiom:

$$\langle \mathbf{u},\ \mathbf{u}\rangle \geq 0 \text{ and } \langle \mathbf{u},\ \mathbf{u}\rangle = 0 \ \Leftrightarrow\ \mathbf{u}=\mathbf{O}$$

Show that the given function fails this axiom.

In this space the length or norm $\|\mathbf{u}\|$ of a vector **u** is given by

$$\|\mathbf{u}\|^2 = |\langle \mathbf{u},\ \mathbf{u}\rangle|$$

The distance function is given by

$$d(\mathbf{u},\ \mathbf{v}) = \left\| \begin{pmatrix}u_1-v_1 & u_2-v_2 & u_3-v_3 & u_4-v_4\end{pmatrix}^T \right\|$$

where **u** and **v** is defined as in (*). Find $d(\mathbf{u},\ \mathbf{v})$ for

$$\mathbf{u}=\begin{pmatrix}1 & 2 & 3 & 4\end{pmatrix}^T,\ \mathbf{v}=\begin{pmatrix}5 & 6 & 7 & 8\end{pmatrix}^T$$

15. Let $P_2(x)$ be the vector space of polynomials of degree 2 or less. The following is an inner product on $P_2(x)$:

$$\langle f,\ g\rangle = \int_0^1 f(x)g(x)dx$$

Starting with the standard basis $B=\{1,\ x,\ x^2\}$ find an orthonormal basis for $P_2(x)$.

16. Consider the vector space P_3. Determine the next Legendre polynomial – a cubic polynomial in P_3 which is orthogonal to $1,\ x,$ and $x^2-\frac{1}{3}$ with respect to the inner product given by

$$\langle p,\ q\rangle = \int_{-1}^1 p(x)q(x)$$

17. Let $\mathbf{u}=\begin{pmatrix}1\\2\\1\end{pmatrix}$, $\mathbf{v}=\begin{pmatrix}2\\1\\-4\end{pmatrix}$, $\mathbf{w}=\begin{pmatrix}3\\-2\\1\end{pmatrix}$ and $\mathbf{x}=\begin{pmatrix}9\\-3\\9\end{pmatrix}$.

(i) Show that $\{\mathbf{u},\ \mathbf{v},\ \mathbf{w}\}$ is an orthogonal basis for $\mathbb{R}^3$.

(ii) Write **x** as a linear combination of the vectors **u**, **v** and **w**.

(iii) Scalars $c_1,\ c_2$ and c_3 are given by the formula:

$$c_1 = \frac{\mathbf{x}\cdot\mathbf{u}}{\|\mathbf{u}\|^2},\ c_2 = \frac{\mathbf{x}\cdot\mathbf{v}}{\|\mathbf{v}\|^2} \text{ and } c_3 = \frac{\mathbf{x}\cdot\mathbf{w}}{\|\mathbf{w}\|^2}$$

Determine $c_1,\ c_2$ and c_3. *What do you notice about your result?*

18. Let **u** and **v** be orthogonal non-zero vectors in a vector space and *k* be any scalar. Find the first error, if any, in the following derivation:

$$\begin{aligned}\langle \mathbf{u}-k\mathbf{v},\ \mathbf{u}-k\mathbf{v}\rangle &= \|\mathbf{u}\|^2 - 2k\langle \mathbf{u},\ \mathbf{v}\rangle + k\|\mathbf{v}\|^2 \\ &= \|\mathbf{u}\|^2 + k\|\mathbf{v}\|^2 \\ &\geq 0 \quad \Rightarrow \quad k \geq -\frac{\|\mathbf{u}\|^2}{\|\mathbf{v}\|^2}\end{aligned}$$

19. Let $\mathbf{Q} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an orthogonal matrix. Show that

$$(a+b)^2 + (c+d)^2 = 2$$

20. Let $\mathbf{u}$ and **v** be vectors in a vector space *V*. Find the first error, if any, in the following derivation:

$$\begin{aligned}\langle \mathbf{u}+2\mathbf{v},\ \mathbf{u}+2\mathbf{v}\rangle &= \langle \mathbf{u},\ \mathbf{u}\rangle + \langle \mathbf{u},\ 2\mathbf{v}\rangle + \langle 2\mathbf{v},\ \mathbf{u}\rangle + \langle 2\mathbf{v},\ 2\mathbf{v}\rangle \\ &= \|\mathbf{u}\|^2 + 2\langle \mathbf{u},\ 2\mathbf{v}\rangle + 2\|\mathbf{v}\|^2\end{aligned}$$

21. Prove that if matrix **A** is orthogonal then matrix $\mathbf{A}^\mathrm{T}$ is also orthogonal.

22. Let **A** be an anti-symmetric matrix which means $\mathbf{A}^T = -\mathbf{A}$. Prove that the matrix **Q** given by:

$$\mathbf{Q} = (\mathbf{I}-\mathbf{A})(\mathbf{I}+\mathbf{A})^{-1}$$

is an orthogonal matrix.

23. Let $\mathbf{A} = \begin{pmatrix} a & b \\ -b & c \end{pmatrix}$ where $a > 0$ and $c > 0$.

(i) Show that $\langle \mathbf{Ax},\ \mathbf{x}\rangle \geq 0$ where $\mathbf{x} \in \mathbb{R}^2$.

(ii) Show that $\langle \mathbf{Ax},\ \mathbf{x}\rangle = 0 \Leftrightarrow \mathbf{x} = \mathbf{O}$.

24. Find vectors to complete the following family to form an orthonormal basis for $\mathbb{R}^4$:

$$\mathbf{u} = (1/2 \quad 1/2 \quad 1/2 \quad 1/2)^T, \ \mathbf{v} = (1/2 \quad 1/2 \quad -1/2 \quad -1/2)^T$$

25. Let *S* be the subspace of $\mathbb{R}^4$ spanned by the vectors

$$\mathbf{u} = (1 \quad 0 \quad -1 \quad 2)^T, \ \mathbf{v} = (-1 \quad 1 \quad 1 \quad 0)^T$$

Find a basis for the subspace $S^\perp$ which contains vectors that are orthogonal to the vectors in *S*. Also find an orthonormal basis for $S^\perp$.

26. Let **A** and **B** be orthogonal matrices. Prove that
(a) **AB** is an orthogonal matrix.
(b) $\mathbf{A}^{-1}$ is an orthogonal matrix.

27. The first two Laguerre polynomials are given by $p_0(x) = 1, \ p_1(x) = 1 - x$. By using the inner product

$$\langle f, \ g \rangle = \int_0^\infty f(x) g(x) e^{-x} \mathrm{d}x$$

show that $\{p_0, \ p_1\}$ is an orthonormal set of vectors.

28. Let **u** and **v** be non – zero vectors in a vector space *V*.

In signal processing and radar signal analysis the expression $\left| \left\langle \frac{\mathbf{u}}{\|\mathbf{u}\|}, \ \frac{\mathbf{v}}{\|\mathbf{v}\|} \right\rangle \right|$ measures the degree to which two signals **u** and **v** are similar. A value close to zero means the two signals **u** and **v** are very different in shape whilst a value close to one means they are very similar.

Show that $0 \le \left| \left\langle \frac{\mathbf{u}}{\|\mathbf{u}\|}, \ \frac{\mathbf{v}}{\|\mathbf{v}\|} \right\rangle \right| \le 1$.

Brief Solution

1. $\left\{\mathbf{p}_1=\begin{pmatrix}1\\0\end{pmatrix},\ \hat{\mathbf{p}}_2=\begin{pmatrix}0\\1\end{pmatrix}\right\}$, $\begin{pmatrix}1\\1\end{pmatrix}=-998\begin{pmatrix}1\\0\end{pmatrix}+1000\begin{pmatrix}0.999\\0.001\end{pmatrix}$

3. (a) $\sqrt{30}$ (b) $19.62\ (2\text{ dp})$

4. $\begin{pmatrix}\cos(\text{¯}) & \sin(\text{¯})\\ -\sin(\text{¯}) & \cos(\text{¯})\end{pmatrix}$

5. (a) $\begin{pmatrix}1\\1\\-1\end{pmatrix}$ (b) $\begin{pmatrix}-1\\1\\-1\end{pmatrix}$ (c) $\begin{pmatrix}-1\\1\\1\end{pmatrix}$

6. $\left\{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix},\ \begin{pmatrix}0\\1\\0\end{pmatrix},\ \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix}\right\}$

7. $\left\{\frac{1}{\sqrt{2}},\ \sqrt{\frac{3}{2}}x\right\}$

8. $\mathbf{A}^{-1}=\mathbf{A}^T$

9. Error is in the first line.

10. (i) $\hat{\mathbf{u}}=\frac{1}{\sqrt{a^2+b^2}}\begin{pmatrix}a\\b\end{pmatrix},\quad \hat{\mathbf{v}}=\frac{1}{\sqrt{a^2+b^2}}\begin{pmatrix}-b\\a\end{pmatrix}$ (ii) $\hat{\mathbf{u}}=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix},\quad \hat{\mathbf{v}}=\frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\end{pmatrix}$

11. $\mathbf{Q}=\begin{pmatrix}\mathbf{p}_1 & \hat{\mathbf{p}}_2 & \hat{\mathbf{p}}_3\end{pmatrix}=\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix},\quad \mathbf{A}=\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}1&2&4\\0&3&6\\0&0&5\end{pmatrix}$

12. (i) $\left\{\frac{1}{\sqrt{30}}\begin{pmatrix}1\\2\\3\\4\end{pmatrix},\ \frac{1}{\sqrt{3930}}\begin{pmatrix}13\\26\\-51\\22\end{pmatrix},\ \frac{1}{\sqrt{17292}}\begin{pmatrix}11\\-109\\-23\\69\end{pmatrix}\right\}$

(ii) $\mathbf{Q}=\begin{pmatrix}1/\sqrt{30} & 13/\sqrt{3930} & 11/\sqrt{17292}\\ 2/\sqrt{30} & 26/\sqrt{3930} & -109/\sqrt{17292}\\ 3/\sqrt{30} & -51/\sqrt{3930} & -23/\sqrt{17292}\\ 4/\sqrt{30} & 22/\sqrt{3930} & 69/\sqrt{17292}\end{pmatrix}$

$\mathbf{R}=\begin{pmatrix}30/\sqrt{30} & 17/\sqrt{30} & 55/\sqrt{30}\\ 0 & 131/\sqrt{3930} & 25/\sqrt{3930}\\ 0 & 0 & 132/\sqrt{17292}\end{pmatrix}$

13. $\left\{\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\0\\1\end{pmatrix}, \frac{1}{\sqrt{6}}\begin{pmatrix}1\\-2\\1\end{pmatrix}\right\}$

14. $\sqrt{32}$

15. $\left\{1,\ \sqrt{12}\left(x-\frac{1}{2}\right),\ \sqrt{180}\left(x^2-x+\frac{1}{6}\right)\right\}$

16. $p(x)=x^3-\frac{3}{5}x$

17. (ii) and (iii) $\mathbf{x}=2\mathbf{u}-\mathbf{v}+3\mathbf{w}=\begin{pmatrix}9 & -3 & 9\end{pmatrix}^T$

18. Error is in the first line.

20. No error.

24. $\widehat{\mathbf{x}}=\frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\\0\\0\end{pmatrix}$ and $\hat{\mathbf{y}}=\frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\-1\\1\end{pmatrix}$

25. $B=\left\{\begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}-2\\-2\\0\\1\end{pmatrix}\right\},\ B_{\text{Orthonormal}}=\left\{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \frac{1}{\sqrt{7}}\begin{pmatrix}-1\\-2\\1\\1\end{pmatrix}\right\}$