Problems on Chapter 3

1. Let $\mathbf{A}=\begin{pmatrix}1&2&3\\1&2&3\\1&2&3\end{pmatrix}$ and $\mathbf{b}=\begin{pmatrix}1\\1\\1\end{pmatrix}$. Determine

(i) rank of **A** (ii) null space of **A** (iii) **x** such that $\mathbf{Ax}=\mathbf{b}$

2. Write down a basis for the vector space M_{33} (3 by 3 matrices). Find $\dim(M_{33})$.

Write the matrix $\mathbf{A}=\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}$ as a linear combination of your chosen basis.

3. Let **A** be a 30 by 25 matrix. What is the largest possible rank of matrix **A**?

4. Find the row space of the matrix $\mathbf{A}=\begin{pmatrix}1&2\\3&4\\5&6\end{pmatrix}$.

5. Consider the linear system:

$$\mathbf{Ax}=\mathbf{b} \quad \text{where } \mathbf{A}=\begin{pmatrix}2&1&1\\1&-1&1\end{pmatrix}$$

(a) (i) Solve the homogeneous linear system; $\mathbf{b}=\mathbf{O}$. Give your homogeneous solution in vector form.

(ii) Find the general solution in vector form for $\mathbf{b}_1=\begin{pmatrix}4\\5\end{pmatrix}$.

(iii) Find the general solution in vector form for $\mathbf{b}_2=\begin{pmatrix}4\\1\end{pmatrix}$.

(iv) Determine the general solution in vector for $\mathbf{b}_3=\begin{pmatrix}4\\9\end{pmatrix}$. Note that $\mathbf{b}_3=2\mathbf{b}_1-\mathbf{b}_2$.

(b) Consider the general consistent linear system $\mathbf{Ax}=\mathbf{b}$ where $\mathbf{A}$ is an *m* by *n* matrix, **x** is an *n* by 1 column vector, **b** is a *m* by 1 column vector. Let $\mathbf{x}_H$ be the homogeneous solution of this system, $\mathbf{p}_1$ and $\mathbf{p}_2$ be particular solutions for $\mathbf{b}_1\neq\mathbf{O}$ and $\mathbf{b}_2\neq\mathbf{O}$ respectively. Prove that the general solution **x** for $\mathbf{Ax}=k\mathbf{b}_1+c\mathbf{b}_2$ where *k* and *c* are scalars is given by

$$\mathbf{x}=\mathbf{x}_H+k\mathbf{p}_1+c\mathbf{p}_2$$

6. Consider the consistent linear system $\mathbf{Ax}=\mathbf{b}$ where **A** is a *m* by *n* matrix and **x**, **b** are appropriate size column vectors. Let $\mathbf{p}_1$ and $\mathbf{p}_2$ be particular solutions of $\mathbf{Ax}=\mathbf{b}_1$ and

$\mathbf{Ax}=\mathbf{b}_2$ respectively and $\mathbf{x}_H$ be the homogeneous solution of the system. Write down the general solutions of the following linear systems:

(a) $\mathbf{Ax}=-6\mathbf{b}_1$

(b) $\mathbf{Ax}=3\mathbf{b}_2$

(c) $\mathbf{Ax}=5\mathbf{b}_1-73\mathbf{b}_2$

7. Consider the matrices $\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}, \begin{pmatrix}2 & 5\\ -1 & 3\end{pmatrix}, \begin{pmatrix}-4 & -11\\ 9 & -1\end{pmatrix}$.

(a) Determine whether these matrices are linearly independent.

(b) Find the dimension of the subspace of M_{22} spanned by these matrices.

8. Determine bases for the row and column space of $\mathbf{A}=\begin{pmatrix}1 & 2 & 3 & 4\\ 5 & 6 & 7 & 8\\ 9 & 10 & 11 & 12\\ 13 & 14 & 15 & 16\end{pmatrix}$.

Solve the homogeneous equation $\mathbf{Ax}=\mathbf{O}$ and find a basis for this solution space.

9. Explain what is meant by the set of vectors $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \cdots,\ \mathbf{v}_n\}$ is linearly dependent.

10. Write the vector $\mathbf{w}=\begin{pmatrix}1\\ 1\end{pmatrix}$ with respect to the basis $\mathbf{u}=\begin{pmatrix}1\\ 0\end{pmatrix}$ and $\mathbf{v}=\begin{pmatrix}0.009\\ 0.001\end{pmatrix}$ of $\mathbb{R}^2$. What are the coordinates of the vector **w** in terms of this basis?

11. (i) Show that the vectors $\mathbf{u}=\begin{pmatrix}1\\ 0\end{pmatrix}$ and $\mathbf{v}=\begin{pmatrix}1\\ 10^{-9}\end{pmatrix}$ form a basis for $\mathbb{R}^2$.

(ii) Write the vector $\mathbf{w}=\begin{pmatrix}1 & 1\end{pmatrix}^T$ in terms of this basis.

12. Let M_{25} be the vector space of 2 by 5 matrices. Decide whether the set of matrices $S=\left\{\begin{pmatrix}a_{11} & a_{12} & 0 & 1 & a_{15}\\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25}\end{pmatrix}\right\}$ is a subspace of M_{25}. (Give reasons for your answer.)

13. Let M_{nn} be the vector space of *n* by *n* matrices. Show that the set *S* of invertible matrices in M_{nn} is *not* a subspace of M_{nn}.

14. Let $V=F:\left]-\infty,\ \infty\right[$ be the vector space of continuous functions. Show that the set *S* of differentiable functions in *V* is a subspace of *V*.

15. Find a basis for the plane $x-2y-3z=0$ which is shown below:

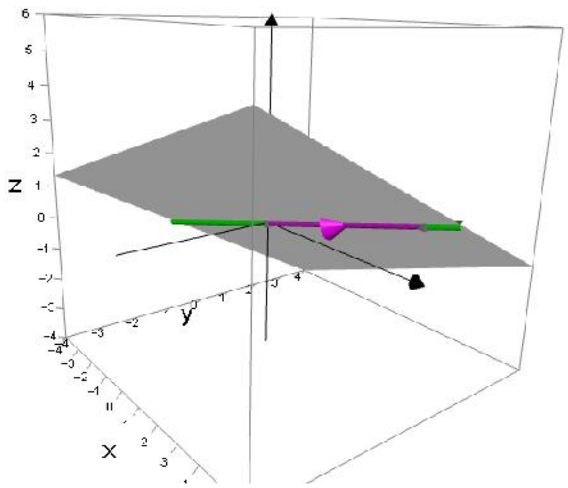

Figure 1

16. Determine the null space, nullity and rank of the matrix $\mathbf{A}=\begin{pmatrix} 3 & -6 & 2 & -1 \\ -2 & 4 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 1 & -6 & 1 & 0 \end{pmatrix}$.

Solve the linear system $\mathbf{Ax}=\mathbf{b}$ where $\mathbf{b}=\begin{pmatrix}-1 & 3 & 1 & 0\end{pmatrix}^T$.

17. Is the set of *x* and *y* axes a vector space of $\mathbb{R}^2$?

18. Is the line $y=\sqrt{2}x$ a vector space of $\mathbb{R}^2$?

19. Is the line $y=\sqrt{2}x+2$ a vector space of $\mathbb{R}^2$?

20. A subspace *S* of $\mathbb{R}^4$ is spanned by the vectors $\mathbf{u}=\begin{pmatrix}2 & -1 & 0 & 1\end{pmatrix}^T$, $\mathbf{v}=\begin{pmatrix}6 & 1 & 4 & -5\end{pmatrix}^T$ and $\mathbf{w}=\begin{pmatrix}28 & -2 & 12 & -10\end{pmatrix}^T$. Find the dimension and a basis of this subspace *S*.

21. Let the column vectors of matrix **A** be linearly independent. Prove that

$$\mathbf{Ax}=\mathbf{O} \quad \Rightarrow \quad \mathbf{x}=\mathbf{O}$$

22. In $\mathbb{R}^6$, test each of the following families of vectors for linear independence, and in each case find a basis for the subspace which they span. Also determine the dimension of the subspace.

(a) $\mathbf{u}=\begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6\end{pmatrix}^T$, $\mathbf{v}=\begin{pmatrix}1 & -3 & 8 & -7 & 1 & 2\end{pmatrix}^T$, $\mathbf{w}=\begin{pmatrix}5 & -5 & 30 & -13 & 13 & 18\end{pmatrix}^T$

(b) $\mathbf{u}=\begin{pmatrix}2 & 1 & 3 & -1 & 4 & -1\end{pmatrix}^T$, $\mathbf{v}=\begin{pmatrix}1 & -1 & 2 & -2 & 3 & -3\end{pmatrix}^T$, $\mathbf{w}=\begin{pmatrix}5 & 5 & 5 & 5 & 5 & 5\end{pmatrix}^T$

23. Let P_2 be the vector space of polynomials of degree 2 or less. Is the set

$$\left\{1-x,\ 1-x^2,\ 1+x+x^2\right\}$$

a basis for P_2?

24. (i) Test whether the following vectors $\left\{\begin{pmatrix}1\\-3\\1\end{pmatrix}, \begin{pmatrix}2\\1\\-4\end{pmatrix}, \begin{pmatrix}-1\\10\\-7\end{pmatrix}\right\}$ in $\mathbb{R}^3$ are linearly independent.

(ii) Decide whether the following set of vectors $\left\{x^2-3x+1,\ 2x^2+x-4,\ -x^2+10x-7\right\}$ form a basis for polynomials of degree 2 or less, P_2 .

25. A subspace *S* of $\mathbb{R}^3$ consists of all the vectors $\mathbf{x}=\begin{pmatrix}x & y & z\end{pmatrix}^T$ which satisfy the following linear system:

$$\begin{aligned} 3x + y - z &= 0 \\ x - 5y + z &= 0 \end{aligned}$$

(i) Determine **x** in vector form. Find a basis for this subspace *S* and the dimension.

(ii) Find the subspace $S^{\perp}$ of $\mathbb{R}^3$ which consists of the vectors that are orthogonal to the vectors in *S*. Find the dimension of $S^{\perp}$.

What do you notice about the dimension of S and $S^{\perp}$*?*

26. Let *S* and *T* be subspaces of $\mathbb{R}^4$ spanned, respectively, by the vectors:

$$S = Span\left\{\begin{pmatrix}1 & -1 & 2 & -3\end{pmatrix}^T,\ \begin{pmatrix}1 & 1 & 2 & 0\end{pmatrix}^T,\ \begin{pmatrix}3 & -1 & 6 & -6\end{pmatrix}^T\right\}$$

$$T = Span\left\{\begin{pmatrix}0 & -2 & 0 & -3\end{pmatrix}^T,\ \begin{pmatrix}1 & 0 & 1 & 0\end{pmatrix}^T\right\}$$

(i) Find a basis for *S* and *T*.

(ii) Determine a basis for subspace which lies in both *S* and *T*. This subspace is normally denoted by $S \cap T$.

(iii) Determine a basis for the subspace which lies in either *S* or *T*. This subspace is normally denoted by $S+T$ and spanned by the vectors in *S* plus the vectors in *T*.

27. Let $S = span\left\{\mathbf{v}_1,\ \mathbf{v}_2,\ \cdots,\ \mathbf{v}_n\right\}$ be subspace of a vector space *V* and the vectors **v**'s are linearly independent. Let **u** be a vector in *V* but not in *S*. Show that the set of vectors

$$\left\{\mathbf{v}_1,\ \mathbf{v}_2,\ \cdots,\ \mathbf{v}_n,\ \mathbf{u}\right\}$$

is linearly independent.

28. Let V be a real vector space of solutions to the differential equation

$$f''(x)-9f(x)=0$$

Show that the vectors $\{e^{3x},\ e^{-3x}\}$ form a basis for *V*.

[This is useful result in a differential equations course.]

29. Let V be a real vector space of solutions to the differential equation which describes simple harmonic motion given by:

$$f''(x)+4f(x)=0$$

Show that the vectors $\{\sin(2x),\ \cos(2x)\}$ form a basis for *V*.

30. In computer science, coding a message is important. The message is sent by binary which is a sequence of 0's and 1's only and where the arithmetic satisfies

$$0+0=0,\ 1+0=1,\ 1+1=0$$

Let V^n be the set of vectors which are ordered *n*-tuples consisting of 0's and 1's with *n* entries. Let $\mathbf{u}=(u_1 \quad u_2 \quad \cdots \quad u_n)^T$ and $\mathbf{v}=(v_1 \quad v_2 \quad \cdots \quad v_n)^T$ be members of this vector space. The Hamming distance $d(\mathbf{u},\ \mathbf{v})$ is defined as

$$d(\mathbf{u},\ \mathbf{v})=\text{number of entries where } u_i \neq v_i$$

Determine the Hamming distance $d(\mathbf{u},\ \mathbf{v})$ of the following vectors:

(a) $\mathbf{u}=(1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1)^T,\ \mathbf{v}=(0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0)^T$

(b) $\mathbf{u}=(1 \quad 0 \quad 0 \quad 1 \quad 1 \quad 0 \quad 1)^T,\ \mathbf{v}=(1 \quad 0 \quad 1 \quad 0 \quad 1 \quad 0 \quad 1)^T$

31. Let **A** be a *nilpotent matrix* which means that $\mathbf{A}^m=\mathbf{O}$ for some natural number *m* and **A** is a square matrix. Examples are $\begin{pmatrix} 0 & 5 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 5 & 6 & 7 & 0 \end{pmatrix}$.

Let *m* be the smallest positive integer such that $\mathbf{A}^m=\mathbf{O}$ and **u** be a vector in $\mathbb{R}^n$ such that $\mathbf{A}^{m-1}\mathbf{u}\neq\mathbf{O}$. Prove that the vectors

$$\{\mathbf{u},\ \mathbf{Au},\ \mathbf{A}^2\mathbf{u},\ \cdots,\ \mathbf{A}^{m-1}\mathbf{u}\}$$

are linearly independent.

32. Give an example of a basis for each of the following vector spaces:

$$P_3,\ M_{22},\ \mathbb{R}^4$$

33. (a) What does it mean to say that a matrix has rank 1? Construct an example of a 3 by 3 matrix which has rank 1.

(b) What does it mean to say that a matrix is of full rank? Construct an example of a 3 by 3 matrix which is of full rank.

34. Let *V* be an *n* dimensional vector space.
(a) What is the maximum number of linearly independent vectors in *V*?
(b) What is the minimum number of vectors that span *V*?

35. Prove that if **A** is an *m* by *n* matrix such that $n > m$ then the column vectors of matrix **A** are linearly dependent.

36. Prove that if **A** is a non-square matrix then the rows or columns of matrix **A** are linearly dependent.

37. Let P_2 be the vector space of polynomials of degree 2 or less. Show that the following $S = \{2,\ 2x-1,\ x^2+1\}$ is a basis for P_2. Write the vector $\mathbf{p} = x^2 + 1$ in this basis.

38. The Frobenius norm denoted $\| \ \|_F$ of a matrix is defined as the square root of the sum of squared entries of the matrix. We can write this in mathematical notation as

$$\|\mathbf{A}\|_F = \sqrt{a_{11}{}^2 + a_{12}{}^2 + \cdots + a_{mn}{}^2} \quad \text{where } \mathbf{A} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

(i) Determine the Frobenius norm $\| \ \|_F$ of the following matrices:

(a) $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (b) $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ (c) $\mathbf{B} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}$

(d) $\mathbf{C} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 6 & 8 & 10 \\ 3 & 6 & 9 & 12 & 15 \end{pmatrix}$

(ii) Find $\sqrt{trace\left(\mathbf{A}^T\mathbf{A}\right)}$ for each of the matrices in part (i).
(iii) What do you notice about your results to parts (i) and (ii)?
(iv) Prove that for any matrix **A**

$$\|\mathbf{A}\|_F = \sqrt{trace\left(\mathbf{A}^T\mathbf{A}\right)}$$

39. Let M_{22} be the vector space of 2 by 2 matrices with our usual matrix addition and scalar multiplication. Let *S* be a subset of M_{22}, containing matrices of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

(i) Show that the set *S* is a subspace of M_{22}.

(ii) Show that matrices in *S* are commutative.

40. An anti –symmetric (or skew symmetric) matrix is a square matrix **A** such that $\mathbf{A}^T = -\mathbf{A}$. Show that the set of *n* by *n* anti-symmetric matrices are a subspace of M_{nn} (the set of *n* by *n* matrices).

41. Show that the real vector space of all twice differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f''(x) + f(x) = 0$$

has basis $\{\cos(x),\ \sin(x)\}$.

[This is an important result in differential equations.]

Brief Solutions

1. (i) 1 (ii) $N=\left\{s\mathbf{u}+t\mathbf{v}\ \middle|\ \mathbf{u}=\begin{pmatrix}-2\\1\\0\end{pmatrix} \text{ and } \mathbf{v}=\begin{pmatrix}-3\\0\\1\end{pmatrix}\right\}$

(iii) $\mathbf{x}=\begin{pmatrix}1\\0\\0\end{pmatrix}+s\begin{pmatrix}-2\\1\\0\end{pmatrix}+t\begin{pmatrix}-3\\0\\1\end{pmatrix}$

2. $B=\left\{\mathbf{E}_1=\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix},\ \mathbf{E}_2=\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix},\ \mathbf{E}_3=\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix},\ \cdots,\ \mathbf{E}_9=\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}\right\}$

$\mathbf{A}=\mathbf{E}_1+2\mathbf{E}_2+3\mathbf{E}_3+\cdots+9\mathbf{E}_9$. Dimension is 9.

3. 25

4. $\mathbb{R}^2$

5. (a) (i) $\mathbf{x}_H=s\begin{pmatrix}-2\\1\\3\end{pmatrix}$ (ii) $\mathbf{x}=\underbrace{\begin{pmatrix}-1\\0\\6\end{pmatrix}}_{\text{Particular soln}}+\underbrace{s\begin{pmatrix}-2\\1\\3\end{pmatrix}}_{\text{Homogeneous soln}}$ (iii) $\mathbf{x}=\underbrace{\begin{pmatrix}3\\0\\-2\end{pmatrix}}_{\text{Particular soln}}+\underbrace{s\begin{pmatrix}-2\\1\\3\end{pmatrix}}_{\text{Homogeneous soln}}$

(iv) $\mathbf{x}=\underbrace{\begin{pmatrix}-5\\0\\14\end{pmatrix}}_{\text{Particular soln}}+\underbrace{s\begin{pmatrix}-2\\1\\3\end{pmatrix}}_{\text{Homogeneous soln}}$

6. (a) $\mathbf{x}=\mathbf{x}_H-6\mathbf{p}_1$ (b) $\mathbf{x}=\mathbf{x}_H+3\mathbf{p}_2$ (c) $\mathbf{x}=\mathbf{x}_H+5\mathbf{p}_1-73\mathbf{p}_2$

7. (a) dependent (b) 2

8. $B_r=\left\{\begin{pmatrix}1\\2\\3\\4\end{pmatrix},\begin{pmatrix}0\\1\\2\\3\end{pmatrix}\right\}$, $B_c=\left\{\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\begin{pmatrix}2\\1\\0\\0\end{pmatrix}\right\}$. $\mathbf{x}=s\begin{pmatrix}1\\-2\\1\\0\end{pmatrix}+t\begin{pmatrix}2\\-3\\0\\1\end{pmatrix}$, basis $\left\{\begin{pmatrix}1\\-2\\1\\0\end{pmatrix},\begin{pmatrix}2\\-3\\0\\1\end{pmatrix}\right\}$.

10. $\begin{pmatrix}-8\\1000\end{pmatrix}$

11. $1\,000\,000\begin{pmatrix}1\\10^{-6}\end{pmatrix}-999\,999\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix}$

12. No

15. $\left\{t\mathbf{u}+s\mathbf{v}\ \middle|\ \mathbf{u}=\begin{pmatrix}2\\1\\0\end{pmatrix},\ \mathbf{v}=\begin{pmatrix}3\\0\\1\end{pmatrix}\right\}$

16. $rank(\mathbf{A})=3$, $nullity(\mathbf{A})=1$ and $N(\mathbf{A})=\left\{r\begin{pmatrix}1\\0\\-1\\1\end{pmatrix}\right\}$, $\mathbf{x}=\begin{pmatrix}0\\0\\0\\1\end{pmatrix}+r\begin{pmatrix}1\\0\\-1\\1\end{pmatrix}$

17. No
18. Yes
19. No
20. Dimension is 2 and a basis is $\left\{\begin{pmatrix}2\\-1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\\-2\end{pmatrix}\right\}$.

22. (a) $\left\{\mathbf{u}, \begin{pmatrix}0 & -5 & 5 & -11 & -4 & -4\end{pmatrix}^T\right\}$, Dim is 2.

(b) $\left\{\mathbf{u}, \begin{pmatrix}0 & 3 & 4 & 0 & 5 & 1\end{pmatrix}^T, \begin{pmatrix}0 & 2 & -1 & 3 & -2 & 4\end{pmatrix}^T\right\}$, Dim is 3.

23. No
24. (i) Dependent (ii) No
25. (i) $\mathbf{x}=r\begin{pmatrix}1\\1\\4\end{pmatrix}$, $\left\{\begin{pmatrix}1\\1\\4\end{pmatrix}\right\}$, $\dim(S)=1$ (ii) $Span\left\{\begin{pmatrix}3\\1\\-1\end{pmatrix}, \begin{pmatrix}1\\-5\\1\end{pmatrix}\right\}$, $\dim(S^\perp)=2$,

$\dim(S)+\dim(S^\perp)=\dim(\mathbb{R}^3)=3$

26. (i) A basis for *S* is $\left\{\begin{pmatrix}1 & -1 & 2 & -3\end{pmatrix}^T, \begin{pmatrix}0 & 2 & 0 & 3\end{pmatrix}^T\right\}$ and for *T* is

$\left\{\begin{pmatrix}0 & -2 & 0 & -3\end{pmatrix}^T, \begin{pmatrix}1 & 0 & 1 & 0\end{pmatrix}^T\right\}$. (ii) $\left\{\begin{pmatrix}0 & 2 & 0 & 3\end{pmatrix}^T\right\}$

(iii) $\left\{\begin{pmatrix}1 & -1 & 2 & -3\end{pmatrix}^T, \begin{pmatrix}0 & 2 & 0 & 3\end{pmatrix}^T, \begin{pmatrix}1 & 0 & 1 & 0\end{pmatrix}^T\right\}$

30. (a) 7 (b) 2
32. $\{1, x, x^2, x^3\}$ for P_3, $\left\{\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\right\}$ for M_{22} and

$\left\{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right\}$ for $\mathbb{R}^4$

34. (a) *n* (b) *n*
37. $\mathbf{p}=0+0+1(x^2+1)$
38. (i) (a) $\sqrt{2}$ (b) $\sqrt{30}$ (c) 14 (d) $\sqrt{770}$ (ii) Same as part (i).