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$$A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}.$$ Determine the orthogonal matrix Q.

$A^T = A$.

The eigenvalues and eigenvectors are given by:

$$\lambda_1 = -1, \quad \underline{u} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}; \quad \lambda_2 = -1, \quad \underline{v} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}; \quad \lambda_3 = 5, \quad \underline{w} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Checking orthogonality:

$$\underline{u} \cdot \underline{v} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = -4 - 3 + 1 = -6 \neq 0.$$

$\underline{u}$ & $\underline{v}$ are not orthogonal.

Applying the Gram-Schmidt Process (4.16):

Let $$\underline{q}_1 = \underline{u} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}.$$

$$\underline{q}_2 = \underline{v} - \frac{\underline{v} \cdot \underline{q}_1}{\|\underline{q}_1\|^2} \underline{q}_1 \qquad (*)$$

$$\underline{v} \cdot \underline{q}_1 = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = -6$$

$$\|\underline{q}_1\|^2 = \left\| \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \right\|^2 = (-2)^2 + 1^2 + 1^2 = 6$$

substituting into (*) yields:

$$\underline{q}_2 = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} - \frac{-6}{6} \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ \end{pmatrix} \quad \begin{pmatrix} -2 \\ \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2-2 \\ -3+1 \\ 1+1 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

Let $q_2^* = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$.

$$\underline{q}_1 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}, \quad q_2^* = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \quad \underline{w} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\hat{q}_1 = \frac{1}{\sqrt{(-2)^2 + 1^2 + 1^2}} \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{6}} \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

$$\underline{\hat{q}}_2^* = \frac{1}{\sqrt{0^2 + (-1)^2 + 1^2}} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

$$\underline{\hat{w}} = \frac{1}{\sqrt{1^2 + 1^2 + 1^2}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$Q = \begin{pmatrix} \underline{\hat{q}}_1 & \underline{\hat{q}}_2^* & \underline{\hat{w}} \end{pmatrix} = \begin{pmatrix} -2/\sqrt{6} & 0 & 1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

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$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \qquad A^{10} = 2^9 A \quad \text{and} \quad A^m = 2^{m-1} A.$$

The eigenvalues and corresponding e.vectors are given by:

$$\lambda_1 = 0, \ \underline{u} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}; \qquad \lambda_2 = 2, \ \underline{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$A^T = A$.

$$\underline{u} \cdot \underline{v} = 0$$

Normalizing:

$$\hat{\underline{u}} = \frac{1}{\sqrt{1^2 + (-1)^2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\hat{\underline{v}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$Q = (\hat{\underline{u}}, \hat{\underline{v}}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$A^{m} = Q D^{m} Q^T$$

$$D = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, \qquad Q^T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$A^{10} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}^{10} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 2^{10} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \qquad 0 = 0 \times 2^{N}$$

$$= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} 2^{10} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$= \frac{2^{10}}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$= 2^9 \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$= 2^9 \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}}_{= A} = 2^9 A.$$

$$A^m = 2^{m-1} A$$

$$A^m = Q D^m Q^T$$

$$= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 2^m \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$= \frac{2^m}{2} \underline{\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}}$$

$$= 2^{m-1} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \underline{2^{m-1} A}.$$