SECTION 1.6 → Inequalities, modulus function and polynomials

By the end of this section you will be able to

- derive properties of inequalities
- understand the modulus function
- examine polynomials

This is a challenging section because we treat inequalities in an abstract manner and many students have difficulty in relating to the results obtained in this section. Additionally, students apply the rules of inequalities as they would for equality, $=$, because they are comfortable in using them. You need to be a lot more careful with inequalities and *cannot* blindly apply the same rules as you did for equality.

I.6.1 Examples of Inequalities

What does the term 'inequality' mean in the general sense?

We often read in the papers 'inequality of wealth' or 'inequality of women's pay' which implies that the distribution has *not* been equally dispersed.

Generally, inequality is opposite to equality which implies that inequality means *not* equal to. For example we can say 'a is greater than 3' and this is denoted by '$a > 3$' meaning that the real number a is to the right of 3 on the real number line:

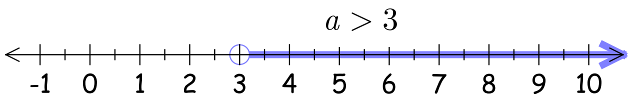

Figure 14

The real number a is less than 3 is denoted by $a < 3$ and means that a is to the left of 3 on the real number line:

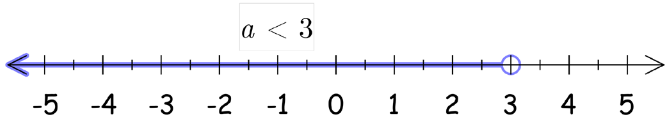

Figure 15

I.6.2 Inequality Properties of Real Numbers

Let a and b be real numbers then *only* one of the following is true:

(i) $a > b$ (a is greater than b).

(ii) $a = b$ (a is equal to b).

(iii) $a < b$ (a is less than b).

Richard Recorde 1512-58 devised the $=$ symbol in 1557. In 1600 Thomas Harriot 1560 – 1621 added the inequality signs $<$ and $>$ and the convention of writing ab for $a \times b$.

The real numbers a and b satisfying $a > b \Leftrightarrow a - b > 0$. For example, in the above case $a > 3 \Leftrightarrow a - 3 > 0$ and $a < 3 \Leftrightarrow a - 3 < 0$.

Also $a > b$ is equivalent to $b < a$. This can be written in mathematical notation as:

$$(a > b) \equiv (b < a).$$

For example $5 > 3$ (5 is greater than 3) is equivalent to $3 < 5$ (3 is less than 5).

What is $\pi < 4$ *equivalent to?*

$$4 > \pi$$

We assume the following properties of inequalities of real numbers:

(I.25) If $a > 0$ and $b > 0$ then $a + b > 0$. *What does this notation mean?*

Adding two positive real numbers gives a positive real number.

(I.26) If $a > 0$ and $b > 0$ then $a \times b = ab > 0$. *What does this notation mean?*

Multiplying two positive real numbers gives a positive real number.

(I.27) If $a \in \mathbb{R}$ then *only* one of the following is true: $a > 0$, $a = 0$ or $-a > 0$.

From these we prove other properties of inequalities of real numbers.

Proposition (I.28). Let a, b and c be real numbers. Then the following properties hold:

(a) If $a > b$ then $a + c > b + c$.

(b) If $a > b$ and $b > c$ then $a > c$.

Proof.

(a) Remember $a > b$ means $a - b > 0$. Therefore

$$\begin{aligned} a - b &= a + \underbrace{c - c}_{=0} - b && \left[\text{Rewriting } \; a - b\right] \\ &= a + c - (b + c) > 0 && \left[\text{Because } \; a - b > 0\right] \end{aligned}$$

From $a + c - (b + c) > 0$ we have $a + c > b + c$. We have proven $a > b \Rightarrow a + c > b + c$.

■

(b) From the given two inequalities, $a > b$ and $b > c$, we have

$$a - b > 0 \text{ and } b - c > 0.$$

Adding these two positive inequalities together gives

$$a \underbrace{- b + b}_{=0} - c > 0$$

$$a - c > 0$$

From $a-c>0$ we have the required result, $a>c$.

■

We can apply the above Proposition (I.28) to concrete examples:

We know $3>2$. Adding $c=5$ does *not* change the inequality:

$$3+5>2+5.$$

Also adding $c=-5$ does *not* change the inequality:

$$3+(-5)>2+(-5).$$

Examples of proposition (I.28) (b) (if $a>b$ and $b>c$ then $a>c$) are

$$3>2 \quad \text{and} \quad 2>1 \quad \Rightarrow \quad 3>1$$

$$-3>-4 \quad \text{and} \quad -4>-5 \quad \Rightarrow \quad -3>-5$$

Proposition (I.29). Let a, b and c be real numbers. We have:

(a) If $a>b$ and $c>0$ then $ac>bc$.

(b) If $a>b$ and $c<0$ then $ac<bc$ [The inequality changes].

Proof.

(a) Recall $a>b$ means $a-b>0$. We are given that $c>0$ and so multiplying these two positive inequalities we have

$$\begin{aligned}(a-b)c &> 0 && \text{[By (I.26)]}\\ ac-bc &> 0 && \text{[Expanding]}\\ ac &> bc\end{aligned}$$

Thus, we have our result.

■

(b) If $c<0$ then $-c>0$ and we are given that $a>b$ so $a-b>0$. Multiplying these two positive inequalities, $a-b>0$ and $-c>0$, we have

$$\begin{aligned}(a-b)(-c) &> 0 && \text{[By (I.26)]}\\ -ac+bc &> 0 && \text{[Expanding]}\\ bc &> ac\end{aligned}$$

Note that $bc>ac$ (bc is greater than ac) is equivalent to $ac<bc$ (ac is less than bc) which is our required result.

■

Notice the result of the above proposition (I.29) (b). If we multiply the inequality $a>b$ by a negative real number such as $c<0$ then it changes the inequality sign to $ac<bc$. A negative multiple changes the inequality sign.

If $a<b$ and $c<0$ then what is the inequality between ac and bc?

$$ac>bc$$

Remember a negative multiple *changes* the inequality sign. Note that inequalities such as $<$ and $>$ do *not* behave like equality, =. You need to be a lot more careful with inequalities as the following application of Proposition (I.29) shows:

$$5 > 2 \text{ and } c = 3 \quad \Rightarrow (5\times 3) > (2\times 3) \qquad [\text{Inequality remains the same}]$$
$$5 > 2 \text{ and } c = -3 \quad \Rightarrow (5\times(-3)) < (2\times(-3)) \qquad [\text{Inequality Changes}]$$
$$-2 < -1 \text{ and } c = -4 \quad \Rightarrow (-2\times(-4)) > (-1\times(-4)) \qquad [\text{Inequality Changes}]$$

Proposition (I.30). Let x be a real number. Then $x^2 \geq 0$.

What can x be?

By (I.27), it can be positive, $x > 0$, or equal to zero, $x = 0$, or negative $x < 0$.

We consider each case.

Proof.

If $x = 0$ then $x^2 = 0^2 = 0 \geq 0$ the result holds.

Now consider the positive case, that is let $x > 0$ then

$$x \times x > 0 \qquad [\text{By (I.26)}]$$
$$x^2 > 0$$

In the last instance we let $x < 0$ [Negative], then multiplying this by the same negative real number, $x < 0$, we change the inequality because of (I.29) (b):

$$x \times x > 0 \times x$$
$$x^2 > 0$$

Combining these three cases we have $x^2 \geq 0$.

■

Proposition (I.31).

(a) $1 > 0$

(b) Let $n \in \mathbb{N}$ then $n > 0$.

Proof.

(a) $1 = 1^2 > 0$ [With $x = 1 \neq 0$ in the previous Proposition (I.30).]

■

(b) *What does $n \in \mathbb{N}$ mean?*

This notation $n \in \mathbb{N}$ means that n is a natural number, 1, 2, 3, 4, … . *How do we prove $n > 0$ for $n \in \mathbb{N}$?*

By using induction because it is a result concerning the natural numbers. The procedure for induction outlined in Section I.4 is to show the result for $n=1$, assume it is true for $n=k$ and then prove the given result for $n=k+1$.

By part (a) we have $1>0$. Assume $k>0$. Required to prove $k+1>0$. We have

$$\begin{aligned} k &> 0 \qquad \left[\text{By Assumption}\right] \\ 1 &> 0 \qquad \left[\text{By (a)}\right] \\ k+1 &> 0 \end{aligned}$$

Hence by induction we have proven our result, if $n \in \mathbb{N}$ then $n>0$.

■

I.6.3 Introduction to the Modulus Function

Definition (I.32). Let x be a real number then the **modulus** of x is denoted by $\left|x\right|$ and is defined as

$$\left|x\right| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

What does this mean?

If x is *positive or zero* then $\left|x\right| = x$ and if x is *negative* then $\left|x\right| = -x$.

By Definition (I.32) we have $\left|5\right| = 5$. *What is the value of* $\left|-5\right|$ *?*

By Definition (I.32) we have $\left|-5\right| = -\left(-5\right) = 5$. *What is the value of* $\left|-1/2\right|$ *?*

Similarly, $\left|-1/2\right| = -\left(-1/2\right) = 1/2$. *What is the value of* $\left|-\pi\right|$ *?*

$$\left|-\pi\right| = -\left(-\pi\right) = \pi.$$

What do you notice about your results?

They are all positive. *What is* $\left|0\right|$ *equal to?*

$$\left|0\right| = 0.$$

What does the graph of $\left|x\right|$ *look like?*

Note that the modulus of negative values of x, $\left|-x\right|$, is positive x and for positive and zero x we have $\left|x\right| = x$. Therefore, the graph of $y = \left|x\right|$ is:

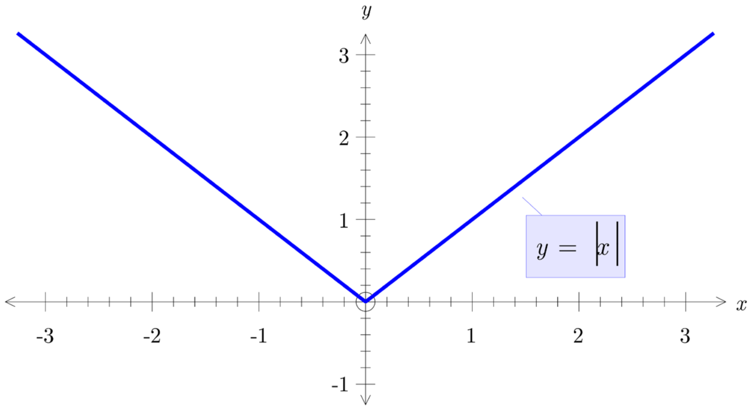

Figure 16

What do you notice about the graph of $y = |x|$ *?*

Symmetrical about the y axis, that is $|-x| = |x|$. *What does the word symmetry mean?*

Symmetry is when two or more parts look alike as they do in the above graph.

Also, the graph of $y = |x|$ is above the x axis and only touches it at $x = 0$. Hence the modulus of a real number is always positive or zero and we can write this as an inequality

(I.33) $$|x| \geq 0$$

What are the values of x *which satisfies* $|x| = 2$ *?*

We have $|x| = 2$ if $x = 2$ or $x = -2$. Therefore $|x| = 2$ has the solutions $x = 2$ or $x = -2$.

The modulus function is also called the **distance** or **absolute function**.

The geometrical interpretation of the modulus function is the distance on the real number line. For example, $|x|$ is the distance from x to zero.

We can visualize $|x| = 2$ as the distance of x from zero is 2 and this is represented by:

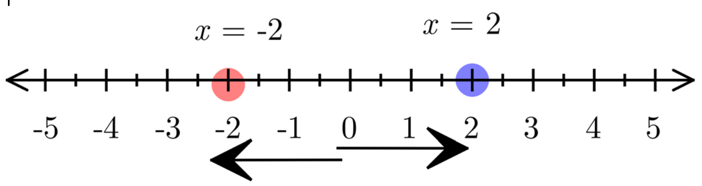

Figure 17

The equation $|x| = a$ has the solution $x = a$ or $x = -a$.

I.6.4 Inequalities of the Modulus Function

Visualizing the modulus function as the distance function is particularly useful for inequalities.

What does $|x| \leq 2$ *represent?*

The distance of x from zero is less than or equal to 2 and can be illustrated as:

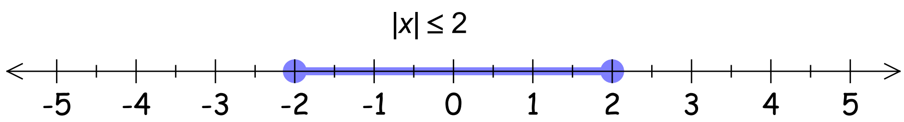

Figure 18

What is the difference between $x \leq 2$ *and* $|x| \leq 2$ *?*

$x \leq 2$ represents all the real numbers less than or equal to 2. The notation $|x| \leq 2$ represents all the real numbers between $x = -2$ to $x = 2$. This is the set of all real numbers between -2 to 2 and is denoted by $\{x \in \mathbb{R} : -2 \leq x \leq 2\}$ which can be written as $-2 \leq x \leq 2$.

What values of x *satisfy* $|x| < 2$ *?*

$$-2 < x < 2$$

We can solve general inequalities such as $|x| < a$ where $a > 0$ by algebraic means.

What does $|x| < a$ *mean?*

The distance x from zero is less than a and can be illustrated on the real line as:

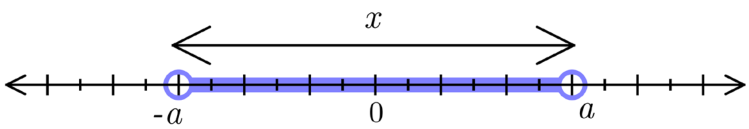

Figure 19

The notation $|x| < a$ means that x lies between $-a$ to a and is denoted by $-a < x < a$ which we can write in symbolic form as

(I.34) $$|x| < a \Leftrightarrow -a < x < a\,.$$

Example 40

Determine the set of real numbers, x, such that $|x - 3| < 5$.

Solution

How can we interpret $|x - 3| < 5$ *as distance on the real number line?*

This inequality, $|x - 3| < 5$, means that the distance from 3 is less than 5 and we can illustrate this by:

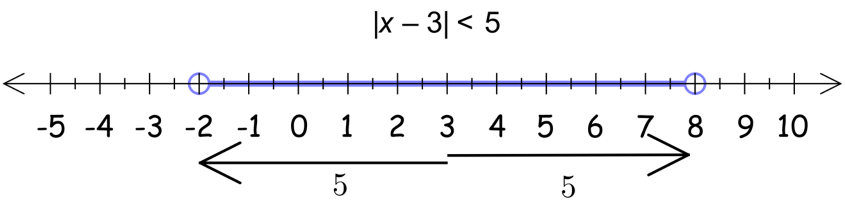

Figure 20

We can also solve this algebraically as follows:

$$\begin{aligned} |x-3| &< 5 \\ -5 < x-3 &< 5 && \text{[By (I.34)]} \\ -5+3 < x &< 5+3 && \text{[Adding 3]} \\ -2 < x &< 8 && \text{[Simplifying]} \end{aligned}$$

Hence all the real numbers between -2 and 8 satisfies the given inequality $|x-3| < 5$.

I.6.5 Properties of the Modulus Function

This subsection is more demanding because the proofs of the results require you to thoroughly understand the definition of the modulus function and inequalities of real numbers as discussed earlier in this section.

Proposition (I.35). For all $x \in \mathbb{R}$ and $y \in \mathbb{R}$ we have

$$|xy| = |x| \times |y| = |x||y|.$$

Proof. We consider the four possible cases:

(1) If both $x \geq 0$ and $y \geq 0$ then $xy \geq 0$ and we have

$$|x||y| = xy = |xy|.$$

(2) If both $x < 0$ and $y < 0$ then $xy > 0$ and we have

$$\begin{aligned} |x||y| &= (-x)(-y) \\ &= xy = |xy| \end{aligned}$$

Remember by the definition of the modulus function if $x < 0$ and $y < 0$ then $|x| = -x$ and $|y| = -y$ respectively.

(3) If $x > 0$ and $y < 0$ then $xy < 0$ and we have

$$\begin{aligned} |x||y| &= x(-y) && \text{[Because } y < 0 \text{ so } |y| = -y\text{]} \\ &= -xy \\ &= |xy| && \text{[Because } xy < 0 \text{ so } |xy| = -xy\text{]} \end{aligned}$$

(4) If $x < 0$ and $y > 0$ then $xy < 0$ and we have

$$\begin{aligned} |x||y| &= (-x)y && \text{[Because } x < 0 \text{ so } |x| = -x\text{]} \\ &= -xy \\ &= |xy| && \text{[Because } xy < 0 \text{ so } |xy| = -xy\text{]} \end{aligned}$$

■

I.6.6 Maximum and Minimum of a Set

Let A be a non – empty set of numbers. Then the minimum m of the set A, denoted $\min A$, satisfies:

I. $m \in A$, that is the m is a member of the set A.

II. For every $a \in A$ we have $m \le a$. This implies that m is less than or equal to every member of A.

Similarly, the maximum M of the set A, denoted $\max A$, satisfies:

I. $M \in A$, that is the M is a member of the set A.

II. For every $a \in A$ we have $M \ge a$. This implies that M is greater than or equal to every member of A.

For example, let $A = \left\{17,\ 4,\ 2019\right\}$ then $\min A = 4$ and $\max A = 2019$.

For another example let B be the set of non – negative integers. Non– negative integer means $\mathbb{N} \cup \left\{0\right\}$, that is all the natural numbers plus zero. Then $\min B = 0$ but it does *not* have a maximum element.

Let C be the set of positive real numbers. Then C has *no* minimum or maximum elements.

I.6.7 Polynomial Expressions

What is a polynomial?

It is an expression of the form

$$c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0$$

where c's are called coefficients and are real numbers. For example

$$2x^4 - x^3 + x^2 + 5x + 7$$

is a polynomial of degree 4 because the highest index of x is 4. A polynomial can be given in factorized form such as

$$y = \left(x-1\right)\left(x-2\right)\left(x-3\right).$$

This is sometimes written in compact form using capital Greek letter pi, $\prod$. The above y can be written as

$$y = \left(x-1\right)\left(x-2\right)\left(x-3\right) = \prod_{j=1}^{3}\left(x-j\right)$$

This is like the summation $\sum$ used for summing terms. For example

$$\prod_{j=2}^{7} j = 2 \times 3 \times 4 \times 5 \times 6 \times 7 = 5040.$$

Example 41

Express $y=\prod_{j=1}^{3}(x-j)$ in $c_n x^n + c_{n-1}x^{n-1}+\cdots+c_1 x + c_0$ form.

Solution

We have

$$\begin{aligned} y=\prod_{j=1}^{3}(x-j) &= (x-1)(x-2)(x-3) \\ &= (x^2-3x+2)(x-3) \\ &= x^3-3x^2+2x-3x^2+9x-6 \\ &= x^3-6x^2+11x-6 \end{aligned}$$

Now we consider quadratic polynomials. *What are quadratic polynomials?*

It is a polynomial of degree 2 and is generally denoted by ax^2+bx+c where a, b and c are real numbers. Now we convert ax^2+bx+c into another format which can be used to find the maximum and minimum values of this quadratic polynomial.

Example 42

Expand $(x+1)^2+5$.

Solution

Opening the brackets and simplifying gives

$$(x+1)^2+5 = x^2+2x+1+5 = x^2+2x+6.$$

Note that x^2+2x+6 can be written as $(x+1)^2+5$. In this part we write the expanded expression x^2+2x+6 as $(x+1)^2+5$. Other examples are

$$x^2+4x+1=(x+2)^2-3$$

$$x^2-5x+2=\left(x-\frac{5}{2}\right)^2-\frac{17}{4}$$

$$x^2+7x+13=\left(x+\frac{7}{2}\right)^2+\frac{3}{4}$$

Do you notice any relationship between the left and right – hand sides?

The number in the bracket on the right – hand side is half the x coefficient on the left.

Considering the first example $x^2+4x+1=(x+2)^2-3$. *The 2 is half of 4 but what about the* -3*?*

Expanding out $(x+2)^2$ gives

$$\left(x+2\right)^2 = x^2 + 4x + 4$$

Subtracting 4 from both sides yields

$$\left(x+2\right)^2 - 4 = x^2 + 4x$$

Now we have $x^2 + 4x$ but we need $x^2 + 4x + 1$, therefore adding 1 to the above gives

$$x^2 + 4x + 1 = \underbrace{\left(x+2\right)^2 - 4}_{=x^2+4x} + 1 \underbrace{=}_{\text{Simplifying}} \left(x+2\right)^2 - 3$$

This process of converting $x^2 + 4x + 1$ to $\left(x+2\right)^2 - 3$ is called completing the square.

Example 43

Complete the square on (a) $x^2 - 8x + 3$ (b) $x^2 - 5x + 3$ (c) $2x^2 - 5x + 3$

Solution

(a) We have

$$x^2 - 8x + 3 = \left(x-4\right)^2 - 4^2 + 3 \underbrace{=}_{\text{Simplifying}} \left(x-4\right)^2 - 13.$$

(b) Similarly, we have

$$\begin{aligned} x^2 - 5x + 3 &= \left(x - \frac{5}{2}\right)^2 - \left(\frac{5}{2}\right)^2 + 3 \\ &= \left(x - \frac{5}{2}\right)^2 - \frac{25}{4} + \frac{12}{4} = \left(x - \frac{5}{2}\right)^2 - \frac{13}{4} \end{aligned}$$

(c) We first take a factor of 2 out:

$$\begin{aligned} 2x^2 - 5x + 3 &= 2\left(x^2 - \frac{5}{2}x + \frac{3}{2}\right) \\ &= 2\left[\left(x - \frac{5}{4}\right)^2 - \left(\frac{5}{4}\right)^2 + \frac{3}{2}\right] \qquad \left[\text{Note } \frac{5}{4} = \frac{1}{2}\left(\frac{5}{2}\right)\right] \\ &= 2\left[\left(x - \frac{5}{4}\right)^2 - \frac{25}{16} + \frac{24}{16}\right] = 2\left[\left(x - \frac{5}{4}\right)^2 - \frac{1}{16}\right] = 2\left(x - \frac{5}{4}\right)^2 - \frac{1}{8} \end{aligned}$$

We can use this completing the square to find the maximum and minimum of a quadratic polynomial as we investigate in the next example.

Example 44

Determine $\min\left\{y \in \mathbb{R} : y = 2x^2 - 5x + 3\right\}$.

Solution

By the previous example part (c) we have

$$2x^2 - 5x + 3 = 2\left(x - \frac{5}{4}\right)^2 - \frac{1}{8}$$

By applying (I.30)

$z^2 \geq 0$

to this $2x^2 - 5x + 3 = 2\left(x - \frac{5}{4}\right)^2 - \frac{1}{8}$ we have

$$2x^2 - 5x + 3 = 2\left(x - \frac{5}{4}\right)^2 - \frac{1}{8} \geq 2(0) - \frac{1}{8} = -\frac{1}{8}$$

Therefore $\min\left\{y \in \mathbb{R} : y = 2x^2 - 5x + 3\right\} = -\frac{1}{8}$. By the way the graph of this is:

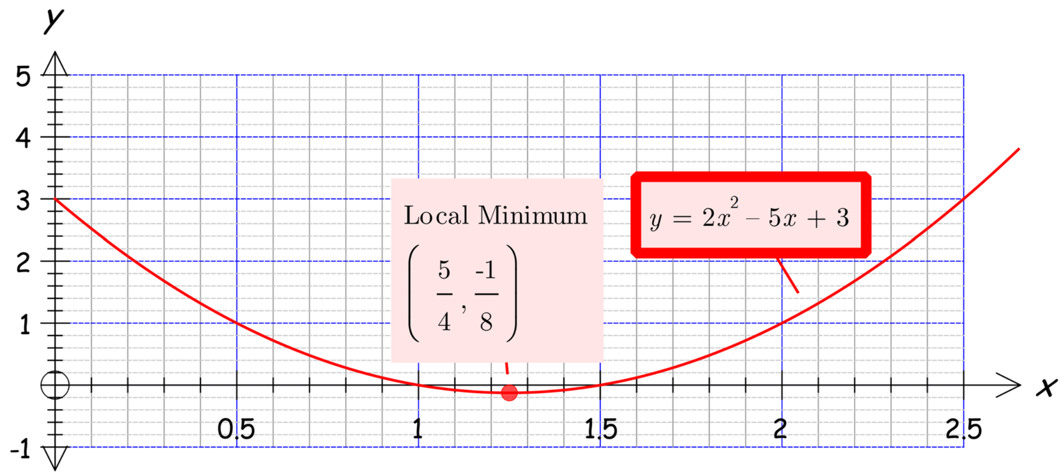

Figure 21

I.6.8 Binomial Expansion

In Exercises I.4 question 18 we proved what is called the binomial expansion:

(I.36) $\left(a+b\right)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \ldots + b^n$

Note that a and b powers are determined by reducing a by an index of 1, giving $a^n, a^{n-1}, a^{n-2}, \cdots, a, a^0$; and increasing b by an index of 1, giving $b^0 = 1, b^1 = b, b^2, \cdots, b^n$.

We can rewrite the binomial expansion as

(I.37) $\left(a+b\right)^n = C_n a^n + C_{n-1}a^{n-1}b + C_{n-2}a^{n-2}b^2 + C_{n-3}a^{n-3}b^3 + \ldots + C_0 b^n$

where C_{n-j} is the coefficient of $a_{n-j}b^j$. We can also find these coefficients by using Pascal's triangle which is given as follows:

Each row begins and ends with 1.

Every number between these 1's is obtained by adding the two numbers in the row above which are immediately to the left and right of it.

1	$n=0$
1 1	$n=1$
1 2 1	$n=2$
1 3 3 1	$n=3$
1 4 6 4 1	$n=4$
1 <u>5</u> 10 10 5 1	$n=5$
1 6 15 <u>20</u> 15 6 1	$n=6$

Figure 22

For example, the 5 underlined in Fig. 22 is determined by adding 1 and 4. Similarly the 20 in the last line is established by summing 10 and 10. Furthermore each of the rows gives the coefficients for the binomial expansion of the corresponding values of n. For example, $n=4$ gives the coefficients 1, 4, 6, 4 and 1. Hence

$$(8q+r)^4=(8q)^4+4(8q)^3r+6(8q)^2r^2+4(8q)r^3+r^4$$

Example 45

Expand $(2q+r)^6$.

Solution

From Fig. 22 we have the coefficients of $n=6$ are 1, 6, 15, 20, 15, 6 and 1. Applying (I.37) we have

$$\begin{aligned}(2q+r)^6&=(2q)^6+6(2q)^5r+15(2q)^4r^2+20(2q)^3r^3+15(2q)^2r^4+6(2q)r^5+r^6\\&=2^6q^6+(6\times 2^5)q^5r+(15\times 2^4)q^4r^2+(20\times 2^3)q^3r^3+(15\times 2^2)q^2r^4+(6\times 2)qr^5+r^6\\&=64q^6+192q^5r+240q^4r^2+160q^3r^3+60q^2r^4+12qr^5+r^6\end{aligned}$$

Summary

In this section we have established properties of inequalities such as:

If $a>b$ and $c>0$ then $ac>bc$.

If $a>b$ and $c<0$ then $ac<bc$. [The inequality changes].

We discussed the modulus function which is the distance or absolute value function.

We converted the general quadratic polynomial by completing the square.

The binomial expansion is given by

$$(a+b)^n=C_na^n+C_{n-1}a^{n-1}b+C_{n-2}a^{n-2}b^2+C_{n-3}a^{n-3}b^3+\ldots+C_0b^n$$

where the coefficients C can be found from Pascal's triangle.