SECTION 1.5 ➡ Joy of Sets

By the end of this section you will be able to

- understand what is meant by a set
- plot Venn diagrams of set operations
- apply the well ordering principle

I.5.1 Introduction to Set Theory

What does the term set mean?

A **set** is a collection of objects and these objects are called **elements** or **members** of the set. The following are examples of sets:

1. The numbers 1, 2, 3 and 4.
2. All the positive odd numbers.
3. European capital cities.
4. The roots of the equation $x^2 - 8x + 7 = 0$.

A set can be described in various ways:

i. By listing *all* the elements of the set. For example, in 1 above the set can be written as $A = \{1,\ 2,\ 3,\ 4\}$. The curly brackets, $\{\ \}$, capture the set and each element in the set is separated by a comma.

ii. By listing the first few elements to give an indication of the pattern of the set. For example, $B = \{1,\ 3,\ 5,\ 7,\ \cdots\}$. Note that the 3 dots (ellipses), $\cdots$, represents the missing members when there is a pattern.

iii. By describing a property of the set such as $C = \{\text{European capital cities}\}$.

iv. By stating a mathematical equation like

$$D = \{x : x^2 - 8x + 7 = 0\}$$

such that

What does the set D mean?

The set D consists of the numbers x such that x satisfies the quadratic equation $x^2 - 8x + 7 = 0$. The colon, :, in the set is read as 'such that'. Hence the set D is the set of numbers x such that $x^2 - 8x + 7 = 0$.

Sets are normally denoted by capital letters such as $A,\ B,\ C \ldots\ X,\ Y \ldots$ The elements or objects of the set are denoted by lower case letters $a,\ b,\ c \ldots\ x,\ y \ldots$

Example 34

Determine the elements of the set D given above.

Solution

We need to solve the quadratic equation given in the set $D=\left\{x: x^2-8x+7=0\right\}$.

The roots of the quadratic equation can be found by factorizing:

$$\begin{aligned} x^2-8x+7&=0 \\ (x-7)(x-1)&=0 \qquad [\text{Factorizing}] \\ x-7=0 \quad &\text{or} \quad x-1=0 \\ x=7 \quad &\text{or} \quad x=1 \end{aligned}$$

We can write the set D as $D=\left\{1,\ 7\right\}$ but it can also be written as $D=\left\{7,\ 1\right\}$. The order of the elements in a set does *not* matter.

We denote the number 7 is a member of the set D by

$$7 \in D.$$

The symbol $\in$ means 'is a member of'. 2 is *not* a member of this set therefore we denote this by $2 \notin D$ and read it as '2 is *not* a member of the set D'.

In general

$x \in A$ means x is a member of the set A.

What does $x \notin A$ mean?

$x \notin A$ means x is *not* a member of the set A.

Example 35

Let A be the set of *all* even numbers. Write the set A in set notation.

Solution

We can write even numbers as the symbol x such that x is an even number, thus we have $A=\left\{x: x \text{ is an even number}\right\}$.

What is the size of the set A?

By size we mean the number of members of the set. A has infinitely many members, so we say the cardinality of A is infinite.

The number of members in a set is called the **cardinality** of the set.

Definition (I.18). Given a set A the **cardinality** of A, denoted $Card\left\{A\right\}$ or $\left|A\right|$, is defined as the number of elements of the set A.

Can you think of an example of a finite set?

The above set $D=\left\{1,\ 7\right\}$ therefore $Card\left\{D\right\}=2$.

If $A=\left\{a,\ b,\ c,\ d,\ f,\ s,\ z\right\}$ then what is $Card\left\{A\right\}$ equal to?

$$Card\{A\} = |A| = 7 \quad [\text{Because the set } A \text{ has 7 members}].$$

I.5.2 Types of Sets

There may be *no* elements in a set. *What do you think we call a set which has no members?*

The **empty set** or the **null set**. The empty set is normally denoted by $\varnothing$ (The Greek letter phi, pronounced fee). *Can you think of any examples of the empty set?*

$$\{\text{Humans who can walk on water}\}$$

What does the universal set mean?

Universal set is the set of *all* the elements under consideration. For example, if we are discussing prime numbers then the universal set will be the set of *all* prime numbers. The universal set is denoted by U.

There are various types of numbers that we have used throughout our lives, but they have *not* been placed in set form or been given a special symbol. *Can you remember what types of numbers you have used?*

Natural numbers, integers, rational numbers and real numbers. We can give *all* these their own symbol:

$\mathbb{N} =$ the set of all natural numbers 1, 2, 3, 4, … These are sometimes called the counting numbers or positive integers. *What is an integer?*

$\mathbb{Z} =$ the set of all integers $\ldots -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3, \ldots$ This is the set of all whole numbers.

$\mathbb{Q} =$ the set of all rational numbers. These are numbers which can be written as ratios or fractions such as $\frac{2}{3},\ -\frac{5}{2},\ \frac{100}{2},\ 6,\ -\frac{1}{7}$. Note that all the integers are also in this set because numbers like 6 can be written as $\frac{12}{2}$.

Numbers such as $\pi,\ \sqrt{2},\ e$ *cannot* be written as fractions so these are not rational numbers. These are examples of **irrational numbers**.

$\mathbb{R} =$ the set of all real numbers. This is the set of all rational and irrational numbers. For example, $\pi,\ \frac{22}{7},\ \sqrt{2},\ \frac{41}{29},\ 5,\ -666,\ 2.333\ldots$ are all members of $\mathbb{R}$. The reals come in two different flavours; rationals and irrationals.

$\mathbb{C} =$ the set of all complex numbers. This set contains all the real numbers as well as numbers such as $\sqrt{-1}$ which is *not* a real number. Complex numbers are normally written as $a + bi$ where i denotes an imaginary number and is equal to $\sqrt{-1}$.

Example 36

Determine the members of the set $A = \left\{x \in \mathbb{N} : \left(x-3\right)\left(2x+1\right) = 0\right\}$.

Solution

The already factorized quadratic produces the solutions:

$$\left(x-3\right)\left(2x+1\right) = 0 \;\Rightarrow\; x = 3 \;\text{ or }\; x = -\frac{1}{2}.$$

Does the set A contain both these elements 3 and $-\frac{1}{2}$?

No, because the set A has the qualification $x \in \mathbb{N}$. *What does this notation $x \in \mathbb{N}$ mean?*

x is a member of the set of natural numbers which means x is a positive whole number. The rational number $-\frac{1}{2}$ is *not* a natural number therefore it *cannot* be a member of the set A. Thus, the set A only has the element 3, that is $A = \left\{3\right\}$.

Example 37

Write the following statements in set notation:

(a) The set of positive real numbers excluding 0.

(b) The set of negative integers.

(c) The set of rational numbers between 0 and 1 (inclusive).

Solution

(a) The set of positive real numbers can be written as a symbol x which represents a real number such that it is greater than 0:

$$\left\{x \in \mathbb{R} : x > 0\right\}.$$

(b) *What is the symbol for the set of integers?*

$\mathbb{Z}$ represents the set of all integers. The set of negative integers can be written as x which is an integer such that it is less than 0:

$$\left\{x \in \mathbb{Z} : x < 0\right\}.$$

(c) *What is the symbol for the set of rationals?*

$\mathbb{Q}$ represents the set of rationals (Q for quotient). The set of rationals between 0 and 1 can be written as:

$$\left\{x \in \mathbb{Q} : 0 \le x \le 1\right\}.$$

I.5.3 Venn Diagrams

Venn diagrams are a graphically way of representing sets. Venn diagrams were introduced by John Venn.

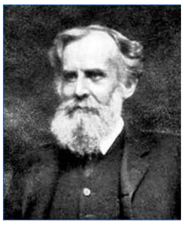

Figure 8 Venn 1834 to 1923

John Venn was born in Hull, England in 1834. His father and grandfather were priests and John was also groomed for a similar post. In 1853 he went to Gonville and Caius College Cambridge and graduated in 1857 becoming fellow of the college. For the next 5 years he went into priesthood and returned to Cambridge in 1862 to teach logic and probability theory.

John Venn is popular known as the person who developed a graphically way to look at sets and this graph become known as a Venn diagram. The sets were represented by oval or circular shape figures but they can be any shape.

It was the Swiss mathematician Euler 1707-83 (pronounced 'oiler') who first discovered Venn diagrams.

Consider the set $A = \left\{x \in \mathbb{Z} : -3 \leq x \leq 2\right\}$. *What are the elements of the set A?*

A is the set of integers which lie between -3 to 2. Thus, the elements are $-3,\ -2,\ -1,\ 0,\ 1$ and 2. The Venn diagram of this looks like:

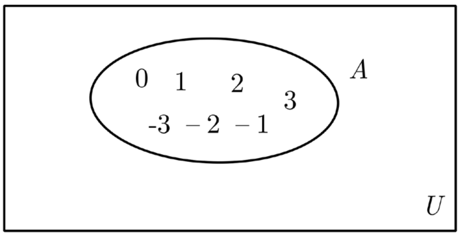

Figure 9

The U in the bottom right hand corner of the rectangle is the universal set which means it includes every element under consideration, which in this case is $U = \mathbb{Z}$. The members of the set A lie within the boundary of the oval shape as shown in Fig. 9.

We can use Venn diagrams to display set operations.

I.5.4 Union and Intersection of Sets

From the age of 7 we have added and multiplied numbers. In a similar fashion we can carry out similar operations on sets. These operations are called **union** and **intersection**.

What is the union of two sets?

The word **union** in everyday language means combining of two or more things. Union of two sets is the combination of all elements in both sets.

Definition (I.19). The **union** of two sets A and B is the set of *all* the elements belonging to set A or set B. The union of two sets A and B is denoted $A \cup B$ and is

$$A \cup B = \{x : x \in A \quad \text{or} \quad x \in B\}$$

In terms of a Venn diagram we can draw $A \cup B$ as:

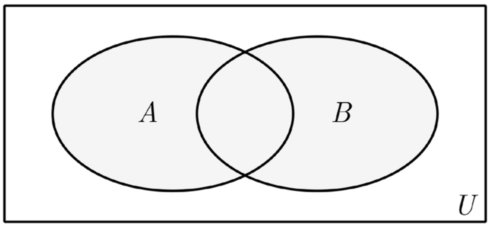

Figure 10 $A \cup B$ (A union B) is shaded

We can also express this $A \cup B$ in terms of mathematical logic discussed in earlier sections. Union $\cup$ is similar to the 'or' symbol which is $\vee$, that is we have

$$x \in A \cup B \quad \Leftrightarrow \quad (x \in A) \vee (x \in B)$$

The other operation on sets is intersection. *What does **intersection** mean in everyday language?*

Intersection means crossroads. Intersection of two sets A and B is the set of *all* elements which belong to *both* sets A and B.

Definition (I.20). The **intersection** of two sets A and B, denoted $A \cap B$, is the set of *all* the elements belonging to set A and set B:

$$A \cap B = \{x : x \in A \quad \text{and} \quad x \in B\}.$$

The Venn diagram of $A \cap B$ is:

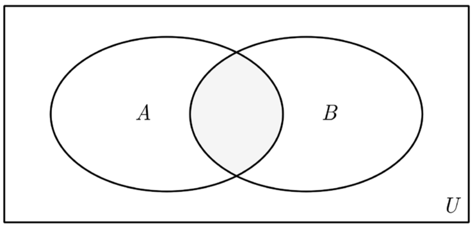

Figure 11 $A \cap B$ (A intersection B) is shaded

Similarly, intersection $\cap$ is similar to the 'and' symbol which is $\wedge$, we have

$$x \in A \cap B \quad \Leftrightarrow \quad (x \in A) \wedge (x \in B).$$

Example 38

Let $A = \{2, \ 3, \ 4, \ 5, \ 6\}$ and $B = \{1, \ 3, \ 7\}$. Determine the sets $A \cup B$ (A union B) and $A \cap B$ (A intersection B). Also draw the Venn diagrams of these sets.

<u>Solution</u>

What does $A \cup B$ mean?

A union B is the set of all elements which are in the set A or B. Thus we have

$$A \cup B = \{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7\}.$$

Which members does the set $A \cap B$ have?

All the members which are common to both the sets A and B.

Only the number 3 belongs to both sets A and B. Therefore

$$A \cap B = \{3\}$$

The Venn diagrams of $A \cup B$ and $A \cap B$ are

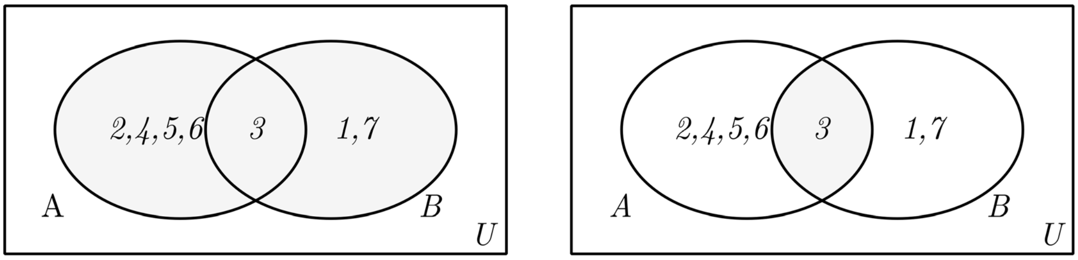

Figure 11 $A \cup B$ is shaded (A or B) $A \cap B$ is shaded (A and B)

I.5.5 Other Set Operations

What does the word ***complement*** *mean in everyday language?*

Complement is something which completes or fills up. In set theory the complement of a set A is the elements which are in the universal set but *not* in set A.

Definition (I.21). The complement of a set A is denoted by A^c and is defined to be

$$A^c = \{x : x \in U,\ \ x \notin A\}.$$

What does the Venn diagram of A^c *look like?*

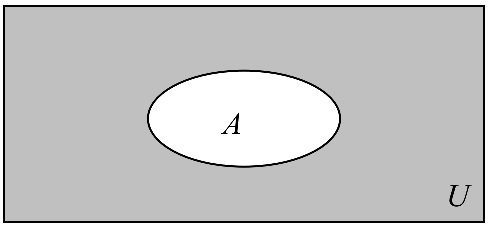

Figure 12 A^c (complement of A) is shaded

Note that $A^C \cup A = U$ where U is the universal set. $A^c \cup A$ fills up U.

We have $x \in A^c \iff x \notin A$. Sometimes A^c is denoted by $\neg A$ or $\sim A$.

Example 39

Let $E = \{2,\ 4,\ 6,\ 8,\ \cdots\}$ and universal set $U = \mathbb{N}$. Determine E^c.

Solution

What does $U = \mathbb{N}$ *mean?*

The universal set U is the set of all the natural numbers 1, 2, 3, 4, … . Note that E is the set of positive even numbers. *What does E^c mean?*

E^c is the set of positive odd natural numbers, that is $E^c = \{1,\ 3,\ 5,\ 7, \cdots\}$.

I.5.6 Introduction to Subsets

What do you think the term ***subset*** *means?*

The prefix 'sub' normally means contained within a system or structure. Thus, subset is a set contained within another set. The Venn diagram of the set A being a subset of the set B is given by:

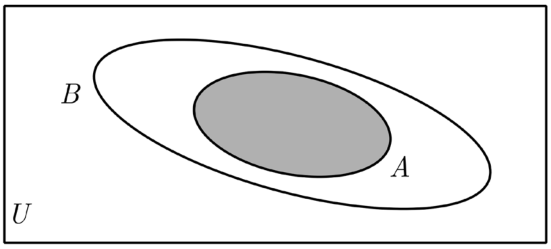

Figure 13 A is a subset of B

What is the definition of a subset?

All the elements of one set are contained within another set. In general, let A and B be two given sets. If every element in set A is also in set B then we say A is a **subset** of B. We also say A is **contained** in B.

How do we denote a subset in mathematical notation?

We denote A is a subset of B by $A \subseteq B$.

An example of a subset is {students} $\subseteq$ {human race}. *Is B a subset of any set in Figure 13?*

The universal set U, that is $B \subseteq U$. Note that set A is also a subset of universal set U. *Is there a set which is the subset of A?*

Yes the empty set $\varnothing$ is a subset of A, that is $\varnothing \subseteq A$. The empty set $\varnothing$ is a subset of *every* set. Also note that A is a subset of itself, that is $A \subseteq A$.

Can you think of any other examples of subsets?

There are an infinitely many examples which you could easily make up such as the following:

1. Let $A = \{\text{Marilyn Munroe, Kim Novak, Michelle Pfeiffer}\}$ and $B = \{\text{women}\}$, then A is a subset of B which we denote by $A \subseteq B$.
2. Let $A = \{\text{Prime Ministers of Britian}\}$ and $B = \{\text{Thatcher, Major, Blair}\}$ then $B \subseteq A$.

3. Let $A = \{a,\ e,\ i,\ o, u\}$ and $B = \{\text{Letters of the alphabet}\}$. Again, we have $A \subseteq B$.

There are many more interesting examples you can create.

The formal definition of subset is:

Definition (I.22). The set A is a **subset** of B if *every* member of set A is also in set B.

I.5.7 Well Ordering Principle

Now we revisit mathematical induction.

Principle of Strong Mathematical Induction (I.23)

For each natural number n, let $P(n)$ be a proposition about n. If $P(n)$ satisfies:

1) $P(n_0)$ is true for base case $n = n_0$ and
2) For all $n \le k$, $P(n)$ is true implies $P(k+1)$ is true.

Then for *all* natural numbers, n, we have $P(n)$ is true.

The difference between strong and ordinary induction can be explained by the ladder analogy:

Ordinary Induction: You are using your present step to prove the next step.

Strong Induction: You have stepped on all the previous steps to get to the current step.

One important property of sets of natural numbers that is often used in proving results is the Well Ordering Principle - WOP:

(I.24) Every non-empty subset of positive integers has a *least* element.

Proof.

We use proof by strong induction.

Let S be a subset of positive integers *without* a least element.

If $1 \in S$ then 1 would be the least element in S which is impossible, so $1 \notin S$.

We assume by strong induction that $2 \notin S,\ 3 \notin S,\ \cdots,\ k \notin S$.

Required to prove that $k+1 \notin S$.

Suppose $k+1 \in S$. However this $k+1 \in S$ *cannot* be the case because then $k+1$ would be the least element of S because $2 \notin S,\ 3 \notin S,\ \cdots,\ k \notin S$ and S has *no* least element.

Therefore, by strong mathematical induction we conclude that S is the empty set, which implies that every non-empty subset of positive integers has a *least* element.

■

The Well Ordering Principle (WOP) is equivalent to Mathematical Induction – you are asked to show this in question 13 of Exercises I.5.

Summary

A set is a collection of objects. The notation $x \in A$ means the element x is a member of the set A.

Let A and B be sets. Then we have the following set operations:

The *union* of two sets is given by $A \cup B = \left\{ x : x \in A \text{ or } x \in B \right\}$.

The *intersection* of two sets is given by $A \cap B = \left\{ x : x \in A \text{ and } x \in B \right\}$.

WOP – every non – empty subset of positive integers has a least element.