Introductory Chapter: Mathematical Logic, Proof and Sets

This chapter is a brief introduction to pure mathematics.

At first you will find this a challenging chapter and *not* the kind of mathematics you would have been familiar with. However, to understand this chapter you need to follow each step at a slower pace and become comfortable with the notation used. We will regularly be using the results of this chapter, so it is important that you understand the material thoroughly.

Some sections will seem abstract and lacking in applications, but the practical use of the theory lies in digital electronics, computer science and artificial intelligence.

The proof part of this chapter is particularly challenging because each proof uses different concepts and it is not a handle turning exercise.

SECTION I.1 ➡ Propositional Logic

By the end of this section you will be able to

- write connectives in symbolic form
- construct truth tables for compound propositions
- show equivalence of propositions using truth tables

I.1.1 Propositions

What does the term proposition mean in mathematics?

It is a statement which has a value of true or false. *Which of the following are propositions?*

(a) Arsenal Football Club won the double in 2002.

(b) $5+3=7$

(c) $x+5=2$

(d) Liz Hurley looks beautiful.

(e) The world ended on 6^{th} June 1984.

(a), (b) and (e) are propositions but (c) and (d) are *not* propositions. *Why not?*

(c) is not a proposition because it contains an unknown, x. Since we do not know the value of x therefore we cannot say whether $x+5=2$ is true or false. 'Liz Hurley looks beautiful' is *not* a proposition because beauty is subjective.

Note that a proposition can be false such as (b) $5+3=7$ and (e) The world ended on 6^{th} June 1984.

We normally denote propositions by the letters P, Q, R, S, ...

Propositions P and Q can only be either true or false. We can write this in a table as:

P	Q
True (T)	True (T)
False (F)	False (F)

Table 1

This kind of table is called a *truth table*.

I.1.2 Negation

What does negation mean?

Generally, means the opposite of something. If P is a proposition then the negation of P is (not$P$) and it is denoted by $\neg P$.

Let P : I weigh more than 75kg. *What is* $\neg P$ *equal to?*

$\neg P$: I do *not* weigh more than 75kg.

Example 1

Write the negations of the following statements:

(1) $a < b$

(2) $2+2 \neq 5$

Solution

(1) $a \geq b$ because the opposite of $a < b$ is a is greater than or equal to b.

(2) $2+2=5$ because the opposite of does not equal $(\neq)$ is equal to $(=)$.

Note the answer to (1) is not $a > b$ but $a \geq b$ because you must cover *all* possibilities. That is P and $\neg P$ (not P) must include *all* possibilities.

What is the truth value of $\neg P$ *if* P *is true?*

$\neg P$ is false.

What is the truth value of $\neg P$ *if* P *is false?*

$\neg P$ is true.

Putting all this together we have the following truth table of $\neg P$:

P	$\neg P$
T	F
F	T

Table 2

You need to remember the truth values of (not P) which has the opposite truth value to P. We use this throughout this chapter.

Example 2

Negate the following proposition:

P : There are integers a and b such that $\frac{a}{b} = \sqrt{2}$.

Solution

Negate means (not$P$) or in symbolic form $\neg P$.

$\neg P$: There are *no* integers a and b such that $\frac{a}{b} = \sqrt{2}$.

I.1.3 And

Two propositions can be combined by the word 'and' to form a compound proposition.

A **compound proposition** is two or more propositions assembled together.

Let P and Q be propositions then 'P and Q' is denoted by

$$P \wedge Q.$$

The compound proposition $P \wedge Q$ is called the **conjunction** of the original propositions.

Example 3

Let P: Grass is blue

Q: Pigs will fly

Form the sentence which describes $P \wedge Q$.

Solution

The notation $P \wedge Q$ means P and Q.

$P \wedge Q$: $\underbrace{\text{Grass is blue}}_{P}$ $\underbrace{\text{and}}_{\wedge}$ $\underbrace{\text{pigs will fly}}_{Q}$.

We can make the truth table for $P \wedge Q$ by first listing all the combination of truth values for P and Q in the first two columns. $P \wedge Q$ (P and Q) is *only* true if both P is true and Q is true otherwise it is false as shown in Table 3:

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Table 3

I.1.4 Or

Two propositions can also be combined by the word 'or'. Let P and Q be propositions, then 'P or Q' is denoted by

$$P \vee Q.$$

A sentence of the form 'P or Q' is called a **disjunction**.

Next, we look at the truth table for $P \vee Q$.

Let P and Q be propositions. The compound proposition $P \vee Q$ (P or Q) is true if either one of P or Q is true. It is *only* false when *both* P is false and Q is false:

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Table 4

Example 4

Let $P: 3 < 4$ and $Q: 4 < 3$. Write out the following:

(a) $P \vee Q$ (b) $\neg P$ (c) $(\neg P) \vee (\neg Q)$

Solution

(a) The notation $P \vee Q$ means P or Q:

$$P \vee Q:\ 3 < 4 \text{ or } 4 < 3.$$

(b) $\neg P$: $3 \geq 4$ (The opposite of $3 < 4$ is $3 \geq 4$).

(c) $(\neg P) \vee (\neg Q)$: $3 \geq 4$ or $4 \geq 3$.

Example 5

Let P: A natural number is prime.

Q: A natural number is composite.

Write out the following:

(a) $P \vee Q$ (b) $(\neg P) \vee (\neg Q)$

Solution

a) The notation $P \vee Q$ means P or Q.

$P \vee Q$: A natural number is either prime or composite.

b) The notation $(\neg P) \vee (\neg Q)$ means $(\text{not } P)$ or $(\text{not } Q)$.

$(\neg P) \vee (\neg Q)$: A natural number is *not* prime nor composite.

A symbol which combines statements is called a connective. For example, $\wedge$, $\vee$ and $\neg$ are connectives.

From the three connectives $\neg$, $\wedge$ and $\vee$ we can construct more complex propositions. These *three* are the basis or fundamental connectives and you need to know their truth values. The remaining section relies on you knowing the truth values of these connectives.

Example 6

Construct the truth table for $\neg(P \vee Q)$.

Solution

$\neg(P \vee Q)$ means the opposite of $P \vee Q$. Hence if $P \vee Q$ is true then $\neg(P \vee Q)$ is false and vice versa. Therefore, the truth value of $\neg(P \vee Q)$ is the opposite of the last column of Table 4:

P	Q	$P \vee Q$	$\neg(P \vee Q)$
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

Table 5

Note that the first three columns are the same as Table 4. The last column gives the truth value of $\neg(P \vee Q)$.

Example 7

Construct the truth table for $(\neg P) \wedge (\neg Q)$.

Solution

Remember $\neg P$ will have truth value opposite to P and similarly $\neg Q$ will have truth value opposite to Q. *How do we find the truth values of* $(\neg P) \wedge (\neg Q)$*?*

Remember the connectives $\wedge$ (and) only gives true if *both* $\neg P$ and $\neg Q$ are true else it is false. We obtain

P	Q	$\neg P$	$\neg Q$	$(\neg P) \wedge (\neg Q)$
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Table 6

What do you notice about the truth values of $\neg(P \vee Q)$ *and* $(\neg P) \wedge (\neg Q)$ *?*

Inspect the last columns of Table 5 and Table 6. Clearly $\neg(P \vee Q)$ and $(\neg P) \wedge (\neg Q)$ have the same truth values. We say the compound propositions $\neg(P \vee Q)$ and $(\neg P) \wedge (\neg Q)$ are **logically equivalent** or just equivalent and it is denoted by $\equiv$.

We write this equivalence as

$$\neg(P \vee Q) \equiv (\neg P) \wedge (\neg Q).$$

Propositions are (logically) equivalent if they have the same truth value for every combination.

Example 8

Show that $P \vee (Q \wedge R)$ and $(P \vee Q) \wedge (P \vee R)$ are logically equivalent.

Solution

We construct the truth table for these propositions.

For the three propositions P, Q and R we need 8 rows covering *all* the possible truth values as shown in the first three columns of Table 7.

Col 1	Col 2	Col 3	Col 4	Col 5	Col 6	Col 7	Col 8
P	Q	R	$Q \wedge R$	$P \vee (Q \wedge R)$	$P \vee Q$	$P \vee R$	$(P \vee Q) \wedge (P \vee R)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
F	T	T	T	T	T	T	T
T	F	F	F	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

Table 7

Since column 5 has the same truth values as column 8 we conclude that

$$P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R) \quad \text{[Equivalent]}.$$

I.1.5 Implication

Let P and Q be two Propositions. The compound statement

'P implies Q'

means 'if P then Q'. Implication is denoted by the symbol $\Rightarrow$. That is

$$P \Rightarrow Q,$$

which says P implies Q.

Bertrand Russell made the following statement regarding implication:

"Pure mathematics consists entirely of assertions to the effect that, if such and such a proposition is true of anything, then such and such another proposition is true of that thing."[1]

Example 9

Let P: I am elected.

Q: I will abolish the death penalty.

Form the sentence that describes

(i) $P \Rightarrow Q$ (ii) $Q \Rightarrow P$

Solution

(i) The notation $P \Rightarrow Q$ means 'if P then Q'.

$$\text{If } \underbrace{\text{I am elected}}_{P} \text{ then } \underbrace{\text{I will abolish the death penalty}}_{Q}.$$

(ii) The notation $Q \Rightarrow P$ means 'if Q then P'.

$$\text{If } \underbrace{\text{I abolish the death penalty}}_{Q} \text{ then } \underbrace{\text{I will be elected}}_{P}.$$

The truth table for P implies Q, $P \Rightarrow Q$, is given by:

	P	Q	$P \Rightarrow Q$
Row 1	T	T	T
Row 2	T	F	F
Row 3	F	T	T
Row 4	F	F	T

Table 8

You might think there is a misprint in Table 8, with regards to the bottom two rows, which is read as 'if P is false' then '$P \Rightarrow Q$' is true, independent of the truth value of Q. There is *no* misprint, this is correct. *How can we justify these statements?*

Consider Example 9 where P and Q were the following propositions:

P : I am elected.

Q : I will abolish the death penalty.

If P is false that is 'I am not elected' then I am under no obligation to abolish the death penalty. Implication is like a contract or a promise.

[1] https://www.goodreads.com/quotes/577891-pure-mathematics-consists-entirely-of-assertions-to-the-effect-that

The only situation when $P \Rightarrow Q$ is false (I have broken my promise) is

'If I am elected then I do *not* abolish the death penalty.'

This situation is represented in Row 2 of Table 8.

In general the implication $P \Rightarrow Q$ is *only* false if P is true and Q is false, otherwise $P \Rightarrow Q$ is true.

Another example is if you mow my lawn, then I will pay you £25. This is only false, if you mow my lawn and I *don't* pay you £25.

The implication connective is very important in mathematics because when proving results, the proof consists of a sequence of true statements connected by implication. A proof starts with a statement which we know is true and ends with a statement that we are required to prove. Each true statement follows from the previous true statement by implication.

Example 10

Let P: ABC is a right-angled triangle with sides a, b and c where $a \le b < c$.

Q : The sides of the triangle ABC satisfy $c^2 = a^2 + b^2$.

Write out the following:

(i) $P \Rightarrow Q$ (ii) $Q \Rightarrow P$

Solution

(i) $P \Rightarrow Q$ means if P then Q.

$P \Rightarrow Q$: If ABC is a right-angled triangle with sides a, b and c where $a \le b < c$ then it satisfies $c^2 = a^2 + b^2$.

(ii) $Q \Rightarrow P$: If the sides of the triangle ABC satisfy $c^2 = a^2 + b^2$ then ABC is a right-angled triangle.

Example 10 is Pythagoras's Theorem.

We can show that $P \Rightarrow Q$ is equivalent to $(\neg P) \vee Q$ that is (not$P$) or Q.

Example 11

Show that

$$(P \Rightarrow Q) \equiv ((\neg P) \vee Q) \qquad \text{[Equivalent]}$$

We have placed brackets on the left and right of the equivalent sign, $\equiv$, so that it becomes easier to visualize the propositions.

Solution

What does equivalence mean in this context?

It means $(\neg P) \vee Q$ and $P \Rightarrow Q$ have the same truth values for all possible combinations of truth values of P and Q.

In the first two left hand columns we list the combinations of truth values of P and Q in Table 9. The truth value of $P \Rightarrow Q$ is given in the last column of the previous Table 8. We can work out the truth value of $(\neg P) \vee Q$. *How?*

First determine the truth values of $\neg P$ (not$P$) and then $(\neg P) \vee Q$. The truth table is:

P	Q	$\neg P$	$(\neg P) \vee Q$	$P \Rightarrow Q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Table 9

The last two columns agree for all possible combinations of truth values therefore they are equivalent, that is

$$(P \Rightarrow Q) \equiv ((\neg P) \vee Q) \qquad \text{[Equivalent]}.$$

We can also state $P \Rightarrow Q$ as follows:

P is a sufficient condition for Q.

This means the condition P is enough for Q to be true.

Q is a necessary condition for P.

We sometimes use these terms instead of $P \Rightarrow Q$.

P implies Q

sufficient necessary

Summary

A proposition is a statement that is true or false.

Compound propositions can be made by the connectives ‘and’, ‘or’, ‘not’, denoted by $\wedge$, $\vee$, $\neg$ respectively.

Implication of two statements, P and Q, represents ‘if P then Q’ and is denoted by $P \Rightarrow Q$.