

Exercises I.6

Brief solutions end of Exercises.

Complete solutions at www.oup.co.uk/companion/NumberTheoryLet a , b , c and d be real numbers.

1. Prove that if $a < b$ then $a - c < b - c$.
2. Prove that if $a < b$ and $b < c$ then $a < c$.
3. Prove that if $a < b$ and $c \leq d$ then $a + c < b + d$.

4. Let x be a real number, prove that

- (i) if $x > 0$ then $\frac{1}{x} > 0$.
- (ii) if $x < 0$ then $\frac{1}{x} < 0$.

5. Let a and b be real numbers, show that

- (a) if $0 < a < b$ then $\frac{1}{a} > \frac{1}{b}$.
- (b) if $b < a < 0$ then $\frac{1}{b} > \frac{1}{a}$.

6. Let $x \in \mathbb{R}$ and $0 < a < b$. Prove that

- (a) $ax^2 \leq bx^2$
- (b) $-ax^2 \geq -bx^2$

7. Let $x \in \mathbb{R}$, show that $x^2 - 2x + 1 \geq 0$.8. Let $x \in \mathbb{R}$, show that $x^2 - 5x + 9 \geq \frac{11}{4}$.9. Prove that $(a + b)^2 \geq 4ab$.10. Let x and y be real numbers, prove that

$$\left[\frac{1}{2}(x + y) \right]^2 \leq \frac{1}{2}(x^2 + y^2)$$

11. Determine the following: $|e|$, $|-e|$, $|\sqrt{2}|$, $|-6-7|$ and $\left|\cos\left(\frac{3\pi}{4}\right)\right|$.

12. Solve the following inequalities:

a. $|x| < 1$ b. $|x| < \pi$ c. $|x-1| < 1$ d. $|x-5| \leq 2$

13. Prove that $|x-y| = |y-x|$.

14. Prove that $\left|\frac{1}{x}\right| = \frac{1}{|x|}$ where $x \neq 0$.

15. Prove that $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$ where $y \neq 0$.

16. Prove that if $n \in \mathbb{N}$ then $\left|\frac{1}{n}\right| = \frac{1}{n}$.

17. Let n be a natural number, *disprove* that $(n+1)^2 \geq 2n^2$.

(Challenge: Prove that $(n+1)^2 \leq 2n^2$ for $n \geq 3$).

18. Evaluate $\prod_{j=1}^5 \left(\frac{2j-1}{2j}\right) \left(\frac{2j+1}{2j}\right)$. Also determine $\left|\frac{2}{\pi} - \prod_{j=1}^5 \left(\frac{2j-1}{2j}\right) \left(\frac{2j+1}{2j}\right)\right|$. Give your answer correct to 3sf.

Actually $\prod_{j=1}^{\infty} \left(\frac{2j-1}{2j}\right) \left(\frac{2j+1}{2j}\right) = \frac{2}{\pi}$.

19. Complete the square on the following quadratic polynomials:

(i) (a) $x^2 - 4x + 3$ (b) $x^2 + 7x + 1$ (c) $x^2 + 3x + 5$ (d) $x^2 + 11x - 6$
 (e) $9 + 8x - x^2$ (f) $3x^2 + 7x + 1$ (g) $11x^2 - 5x - 7$ (h) $\frac{8}{7} - \frac{7}{5}x - \frac{1}{2}x^2$

Test yourself using the following online questions at

[Test on Completing the square](#)

(ii) Determine the following:

- (a) $\min\{y \in \mathbb{R} : y = x^2 - 4x + 3\}$ (b) $\min\{y \in \mathbb{R} : y = x^2 + 7x + 1\}$
 (c) $\min\{y \in \mathbb{R} : y = x^2 + 3x + 5\}$ (d) $\min\{y \in \mathbb{R} : y = x^2 + 11x - 6\}$
 (e) $\max\{y \in \mathbb{R} : y = 9 + 8x - x^2\}$ (f) $\min\{y \in \mathbb{R} : y = 3x^2 + 7x + 1\}$
 (g) $\min\{y \in \mathbb{R} : y = 11x^2 - 5x - 7\}$ (h) $\max\left\{y \in \mathbb{R} : y = \frac{8}{7} - \frac{7}{5}x - \frac{1}{2}x^2\right\}$

20. Expand the following:

- (a) $(1 + x)^5$ (b) $(2 + x)^3$ (c) $(5 - x)^4$ (d) $(3 + 2x)^5$ (e) $(5 + x)^4$
 (f) $(2 + 3x)^5$ (g) $(4 - 3x)^6$ (h) $(2x - y)^4$ (i) $\left(x + \frac{1}{x}\right)^5$ (j) $\left(x - \frac{1}{x}\right)^4$
 (k) $(1 + x^2)^7$ (l) $\left(\frac{w}{4} - \frac{x}{3}\right)^7$

Test yourself using the following online questions:

[Test on the Binomial Expansion using Pascal's triangle](#)

Brief Solutions to Exercise I.6

11. $e, e, \sqrt{2}, 13$ and $\frac{1}{\sqrt{2}}$
12. a. $-1 < x < 1$ b. $-\pi < x < \pi$ c. $0 < x < 2$ d. $3 < x < 7$
18. 0.666 and 0.0296
19. (i) (a) $(x - 2)^2 - 1$ (b) $\left(x + \frac{7}{2}\right)^2 - \frac{45}{4}$ (c) $\left(x + \frac{3}{2}\right)^2 + \frac{11}{4}$ (d) $\left(x + \frac{11}{2}\right)^2 - \frac{145}{4}$
 (e) $25 - (x - 4)^2$ (f) $3\left(x + \frac{7}{6}\right)^2 - \frac{37}{12}$ (g) $11\left(x - \frac{5}{22}\right)^2 - \frac{333}{44}$
 (h) $\frac{743}{350} - \frac{1}{2}\left(x + \frac{7}{5}\right)^2$
- (ii) (a) -1 (b) $-\frac{45}{4}$ (c) $\frac{11}{4}$ (d) $-\frac{145}{4}$ (e) 25 (f) $-\frac{37}{12}$
 (g) $-\frac{333}{44}$ (h) $\frac{743}{350}$
20. (a) $1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$
 (b) $8 + 12x + 6x^2 + x^3$
 (c) $625 - 500x + 150x^2 - 20x^3 + x^4$
 (d) $243 + 810x + 1080x^2 + 720x^3 + 240x^4 + 32x^5$

(e) $625 + 500x + 150x^2 + 20x^3 + x^4$

(f) $32 + 240x + 720x^2 + 1080x^3 + 810x^4 + 243x^5$

(g) $4\,096 - 18\,432x + 34\,560x^2 - 34\,560x^3 + 19\,440x^4 - 5\,832x^5 + 729x^6$

(h) $16x^4 - 32x^3y + 24x^2y^2 - 8xy^3 + y^4$

(i) $x^5 + 5x^3 + 10x + \frac{10}{x} + \frac{5}{x^3} + \frac{1}{x^5}$

(j) $x^4 - 4x^2 + 6 - \frac{4}{x^2} + \frac{1}{x^4}$

(k) $1 + 7x^2 + 21x^4 + 35x^6 + 35x^8 + 21x^{10} + 7x^{12} + x^{14}$

(l) $\frac{w^7}{16384} - \frac{7}{12288}w^6x + \frac{7}{3072}w^5x^2 - \frac{35}{6912}w^4x^3 + \frac{35}{5184}w^3x^4 - \frac{7}{1296}w^2x^5 + \frac{7}{2916}wx^6 - \frac{1}{2187}x^7$