Complete solutions to Exercises I.1

1. The following are propositions:
(a), (b) and (c). Only (a) is true.

2. (i) Man cannot be pregnant.
(ii) Grass is not green.
(iii) Lecturers annual salary is less than or equal to £45 000.
(iv) There are *no* integers a and b such that $\dfrac{a}{b} = \pi$.
(v) There are *no* integers a and b such that $\dfrac{a}{b} = e$.

3. If $x^2 - 9 = 0$ then $x^2 = 9$. If $x^2 = 9$ then $x = \sqrt{9}$. If $x = \sqrt{9}$ then $x = \pm 3$.

4. (i) If $x < 3$ then $x^2 < 9$.
(ii) If $x^2 < 9$ then $x < 3$.
Yes, both propositions are true.

5. (i) If ABC is an equilateral triangle then all the angles inside the triangle ABC are equal.
(ii) If all the angles inside the triangle ABC are equal then ABC is an equilateral triangle.
Both of these are true.

6. (i) If n is prime then $2^n - 1$ is prime.
(ii) If $2^n - 1$ is prime then n is prime.
Part (i) is false because 11 is prime but

$$2^{11} - 1 = 2047 = 23 \times 89 .$$

Part (ii) is true.

7. The truth table is given by:

Q	P	$Q \vee P$
T	T	T
T	F	T
F	T	T
F	F	F

By comparing with the truth table for $P \vee Q$ we have

$$\big(P \vee Q\big) \equiv \big(Q \vee P\big) \qquad \text{[Equivalent]}.$$

8. Truth table is

Q	P	$Q \wedge P$
T	T	T
T	F	F
F	T	F
F	F	F

By comparing with the truth table for $P \wedge Q$ we have

$$\big(P \wedge Q\big) \equiv \big(Q \wedge P\big) \quad \text{[Equivalent]}.$$

9. (a)

P	$\neg P$	$\big(\neg P\big) \wedge P$
T	F	F
F	T	F

Clearly (not P) and P is going to give you false. That is

$$\big(\neg P\big) \wedge P \equiv F\,.$$

(b)

P	$\neg P$	$\big(\neg P\big) \vee P$
T	F	T
F	T	T

Clearly (not P) or P is going to give you true (T). Such a proposition is called a tautology which we discuss in the next section:

$$\big(\neg P\big) \vee P \equiv T\,.$$

(c)

P	$\neg P$	$\neg\big(\neg P\big)$
T	F	T
F	T	F

Clearly $\neg\big(\neg P\big) \equiv P$.

10. (a) $\neg\Big(\neg\big(\neg P\big)\Big) \equiv \neg P$

(b) $P \wedge P \equiv P$

(c) $P \vee P \equiv P$

(d) $\big(\neg P\big) \wedge \big(\neg P\big) \equiv \neg P$ [From 10(b).]

11. The truth table is

P	Q	$P \wedge Q$	$\neg\big(P \wedge Q\big)$	$\neg P$	$\neg Q$	$\big(\neg P\big) \vee \big(\neg Q\big)$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

Since the shaded columns are the same we conclude that

$$\neg\big(P \wedge Q\big) \equiv \Big[\big(\neg P\big) \vee \big(\neg Q\big)\Big].$$

12. Truth table is:

P	Q	R	$Q \vee R$	$P \wedge \big(Q \vee R\big)$	$P \wedge Q$	$P \wedge R$	$\big(P \wedge Q\big) \vee \big(P \wedge R\big)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T
T	F	T	T	T	F	T	T
F	T	T	T	F	F	F	F
T	F	F	F	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	F	F	F	F	F	F	F

Shaded columns agree therefore

$$P \wedge \big(Q \vee R\big) \equiv \big(P \wedge Q\big) \vee \big(P \wedge R\big).$$

13. The rule is

$$\neg\neg \ldots \neg P \equiv \begin{cases} P & \text{if the number of } \neg \text{ 's is even} \\ \neg P & \text{if the number of } \neg \text{ 's is odd} \end{cases}$$

Since we have 4 $\neg$'s in $\neg\neg\neg\neg P$ therefore using this rule we have

$$\neg\neg\neg\neg P \equiv P$$

which means the cup is full.

14. The truth table is

P	Q	$\neg P$	$\big(\neg P\big) \Rightarrow Q$
T	T	F	T
T	F	F	T
F	T	T	T
F	F	T	F

15. Truth table is:

P	Q	R	$\neg Q$	$P \wedge \big(\neg Q\big)$	$R \wedge \big(\neg R\big)$	$\big(P \wedge \big(\neg Q\big)\big) \Rightarrow \big(R \wedge \big(\neg R\big)\big)$	$P \Rightarrow Q$	$\Big[\big(P \wedge \big(\neg Q\big)\big) \Rightarrow \big(R \wedge \big(\neg R\big)\big)\Big] \Rightarrow \big(P \Rightarrow Q\big)$
T	T	T	F	F	F	T	T	T
T	T	F	F	F	F	T	T	T
T	F	T	T	T	F	F	F	T
F	T	T	F	F	F	T	T	T
T	F	F	T	T	F	F	F	T
F	T	F	F	F	F	T	T	T
F	F	T	T	F	F	T	T	T
F	F	F	T	F	F	T	T	T

The given compound proposition

$\Big[\big(P \wedge \big(\neg Q\big)\big) \Rightarrow \big(R \wedge \big(\neg R\big)\big)\Big] \Rightarrow \big(P \Rightarrow Q\big)$ is always true which means it is a tautology.