Complete Solutions to Exercises I.6

1. *Proof.* Since $a < b$ which means that $a - b < 0$ and so

$$a - b = a \underbrace{-c + c}_{=0} - b$$
$$= a - c - \left(b - c\right) < 0$$

From $a - c - \left(b - c\right) < 0$ we have the required result, $a - c < b - c$.

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2. *Proof.* From the two given inequalities, $a < b$ and $b < c$, we have

$$a - b < 0 \text{ and } b - c < 0 \text{ respectively.}$$

Adding these inequalities, $a - b < 0$ and $b - c < 0$, we have

$$a \underbrace{-b + b}_{=0} - c < 0$$
$$a - c < 0 \quad \text{which implies} \quad a < c$$

Hence, we have $a < c$ which is what we were trying to prove.

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3. *Proof.* From the two given inequalities, $a < b$ and $c \le d$, we have

$$a + c \le a + d < b + d$$

Hence, we have our result, $a + c < b + d$.

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4. (a) *Proof.* We use prove by contradiction. Suppose $\frac{1}{x} < 0$. Multiplying through by $x^2 > 0$ we have

$$\frac{1}{x}\left(x^2\right) < 0\left(x^2\right)$$
$$x < 0$$

$x < 0$ (x is less than 0) is a contradiction because we are given $x > 0$ (x is greater than 0). Hence our supposition $\frac{1}{x} < 0$ is wrong therefore we have our result, $\frac{1}{x} > 0$.

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(b) *Proof.* Suppose $\frac{1}{x} > 0$. Multiplying through by $x^2 > 0$ we have

$$\frac{1}{x}\left(x^2\right) > 0\left(x^2\right)$$
$$x > 0$$

$x > 0$ (x is greater than 0) is a contradiction because we are given $x < 0$ (x is less than 0). Hence our supposition $\frac{1}{x} > 0$ is wrong therefore we have our result, $\frac{1}{x} < 0$.

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5. (a) *Proof.* We use proof by contradiction. Suppose $\frac{1}{a} < \frac{1}{b}$. From this we have

$$\frac{1}{a} - \frac{1}{b} < 0$$
$$b - a < 0 \qquad \left[\text{Multiplying through by } ab\right]$$
$$b < a$$

Remember $b < a \Leftrightarrow a > b$. Contradiction! *How?*

Because we are given $a < b$ (a is less than b) and we have deduced $a > b$ (a is greater than b). Our supposition $\frac{1}{a} < \frac{1}{b}$ must be false. Therefore $\frac{1}{a} \geq \frac{1}{b}$. But $\frac{1}{a} \neq \frac{1}{b}$ because $a \neq b$. Hence, we have the strict inequality, $\frac{1}{a} > \frac{1}{b}$.

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(b) *Proof.* We use proof by contradiction. Suppose $\frac{1}{b} < \frac{1}{a}$. From this we have

$$\frac{1}{b} - \frac{1}{a} < 0$$
$$a - b < 0 \qquad \left[\text{Multiplying through by } ab\right]$$
$$a < b$$

Recall $a < b \Leftrightarrow b > a$. Contradiction! *How?*

Because we are given $b < a$ (b is less than a) and we have deduced $b > a$ (b is greater than a). Our supposition $\frac{1}{b} < \frac{1}{a}$ must be false. Therefore $\frac{1}{b} \geq \frac{1}{a}$. But $\frac{1}{a} \neq \frac{1}{b}$ because $a \neq b$. Hence, we have $\frac{1}{b} > \frac{1}{a}$.

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6. (a) *Proof.* We have $0 < a < b$ and if $x = 0$ then $x^2 = 0$, therefore $ax^2 = bx^2 = 0$. If $x \neq 0$ then by Proposition (I.30) we have $x^2 > 0$ and so $ax^2 < bx^2$. Hence, we have the result that we are trying to prove, $ax^2 \leq bx^2$.

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(b) *Proof.* If $x = 0$ then $x^2 = 0$, therefore $ax^2 = bx^2 = 0$. If $x \neq 0$ then by Proposition (I.30) $x^2 > 0$ and multiplying this by -1 gives

$$\begin{aligned} (-1)x^2 &< (-1)0 \qquad \left[\text{Change Inequality because } -1 < 0\right] \\ -x^2 &< 0 \end{aligned}$$

Multiplying the given inequality, $a < b$, by $-x^2$ we have

$$\begin{aligned} a\left(-x^2\right) &> b\left(-x^2\right) \qquad \left[\text{Change Inequality because } -x^2 < 0\right] \\ -ax^2 &> -bx^2 \qquad \left[\text{Because } a\left(-x^2\right) = -ax^2 \text{ and } b\left(-x^2\right) = -bx^2\right] \end{aligned}$$

which is the result we are trying to prove, $-ax^2 \geq -bx^2$.

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7. *Proof.* We have

$$x^2 - 2x + 1 = (x-1)^2 \geq 0 \qquad \left[\text{By the Proposition (I.30)}\right]$$

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8. *Proof.* We have

$$\begin{aligned} x^2 - 5x + 9 &= \left(x - \frac{5}{2}\right)^2 - \frac{25}{4} + 9 \qquad \left[\text{Completing the Square}\right] \\ &= \left(x - \frac{5}{2}\right)^2 - \frac{25}{4} + \frac{36}{4} \qquad \left[\text{Because } 9 = \frac{36}{4}\right] \\ &= \left(x - \frac{5}{2}\right)^2 + \frac{11}{4} \\ &\geq 0 + \frac{11}{4} = \frac{11}{4} \qquad \left[\text{Because } \left(x - \frac{5}{2}\right)^2 \geq 0\right] \end{aligned}$$

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9. *Proof.* We have

$$\begin{aligned} (a+b)^2 &= a^2 + 2ab + b^2 \\ &= a^2 + b^2 + 2ab \qquad (*) \end{aligned}$$

We need to prove $a^2 + b^2 \geq 2ab$. *How?*

Consider $(a-b)^2$. By Proposition (I.30) we have $(a-b)^2 \geq 0$ and

$$(a-b)^2 = a^2 + b^2 - 2ab \geq 0 \text{ which implies that } a^2 + b^2 \geq 2ab$$

Substituting this inequality, $a^2 + b^2 \geq 2ab$, into (*) gives

$$\begin{aligned} (a+b)^2 &= a^2 + b^2 + 2ab \\ &\geq 2ab + 2ab = 4ab \end{aligned}$$

This, $(a+b)^2 \geq 4ab$, is the required result.

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10. *Proof.* We have

$$\left[\frac{1}{2}\left(x+y\right)\right]^2=\frac{1}{4}\left(x^2+y^2\right)+\frac{1}{2}xy \qquad (\dagger)$$

We are required to prove that $\frac{1}{2}xy\le\frac{1}{4}\left(x^2+y^2\right)$. *How?*

Consider $\left[\frac{1}{2}\left(x-y\right)\right]^2$. By Proposition (I.30) we have $\left[\frac{1}{2}\left(x-y\right)\right]^2\ge 0$ and

$$\begin{aligned}\left[\frac{1}{2}\left(x-y\right)\right]^2&=\frac{1}{4}\left(x^2-2xy+y^2\right) && \left[\text{Expanding}\right]\\&=\frac{1}{4}\left(x^2+y^2\right)-\frac{1}{2}xy\ge 0\\ \frac{1}{4}\left(x^2+y^2\right)&\ge\frac{1}{2}xy\end{aligned}$$

We can write the last line, $\frac{1}{4}\left(x^2+y^2\right)\ge\frac{1}{2}xy$, as $\frac{1}{2}xy\le\frac{1}{4}\left(x^2+y^2\right)$ and substituting this into (†) gives

$$\begin{aligned}\left[\frac{1}{2}\left(x+y\right)\right]^2&=\frac{1}{4}\left(x^2+y^2\right)+\frac{1}{2}xy\\&\le\frac{1}{4}\left(x^2+y^2\right)+\frac{1}{4}\left(x^2+y^2\right)\\&=\frac{1}{2}\left(x^2+y^2\right)\end{aligned}$$

We have proven our result, $\left[\frac{1}{2}\left(x+y\right)\right]^2\le\frac{1}{2}\left(x^2+y^2\right)$.

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11. We use Definition (I.32) which is $\left|x\right|=\begin{cases}x & \text{if } x\ge 0\\ -x & \text{if } x<0\end{cases}$ in each case:

$$\left|e\right|=e,\ \left|-e\right|=-\left(-e\right)=e\,,$$

$$\left|-\sqrt{2}\right|=-\left(-\sqrt{2}\right)=\sqrt{2}\,,$$

$$\left|-6-7\right|=\left|-13\right|=-\left(-13\right)=13\,,$$

$$\left|\cos\left(\frac{3\pi}{4}\right)\right|=\left|-\frac{1}{\sqrt{2}}\right|=-\left(-\frac{1}{\sqrt{2}}\right)=\frac{1}{\sqrt{2}}\,.$$

12. We need to solve inequalities using the modulus function. In each case we use (I.34) $\left|x\right|<a\Leftrightarrow -a<x<a$.

a. We are given $\left|x\right| < 1$ therefore by (I.34) we have

$$\left|x\right| < 1 \;\Leftrightarrow\; -1 < x < 1\,.$$

b. Similarly, we have $\left|x\right| < \pi \;\Leftrightarrow\; -\pi < x < \pi\,.$

c. Now we are given $\left|x-1\right| < 1$ so using (I.34) we have

$$\begin{aligned}\left|x-1\right| < 1 &\Leftrightarrow -1 < x-1 < 1\\ &\Leftrightarrow -1+1 < x < 1+1\\ &\Leftrightarrow \quad 0 < x < 2\end{aligned}$$

d. Similarly we have

$$\begin{aligned}\left|x-5\right| \le 2 &\Leftrightarrow -2 \le x-5 \le 2\\ &\Leftrightarrow -2+5 < x < 2+5\\ &\Leftrightarrow \quad 3 < x < 7\end{aligned}$$

13. We need to prove $\left|x-y\right| = \left|y-x\right|$.

Proof.

We use Proposition (I.35) $\left|xy\right| = \left|x\right|\left|y\right|$ on this. We have

$$\begin{aligned}\left|x-y\right| &= \left|-\left(y-x\right)\right|\\ &= \left|-1\left(y-x\right)\right|\\ &= \left|-1\right|\left|y-x\right| \qquad \left[\text{By (I.35)}\right]\\ &= 1\left|y-x\right| = \left|y-x\right|\end{aligned}$$

Therefore $\left|x-y\right| = \left|y-x\right|$.

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14. We are asked to prove $\left|\frac{1}{x}\right| = \frac{1}{\left|x\right|}$ where $x \ne 0\,.$

Proof.

We use Proposition (I.35) $\left|xy\right| = \left|x\right|\left|y\right|$:

$$\left|\frac{1}{x}\right| = \left|1\times\frac{1}{x}\right| = \left|1\right|\times\left|\frac{1}{x}\right| = 1\times\frac{\left|1\right|}{\left|x\right|} = \frac{1}{\left|x\right|}\,.$$

This completes our proof.

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15. We are asked to prove $\left|\frac{x}{y}\right| = \frac{\left|x\right|}{\left|y\right|}$ where $y \ne 0\,.$

Proof.

We use Proposition (I.35) $\left|xy\right| = \left|x\right|\left|y\right|$ and the result of the previous question we have

$$\left|\frac{x}{y}\right| = \left|x \times \frac{1}{y}\right|$$
$$= \left|x\right| \times \left|\frac{1}{y}\right| = \left|x\right| \times \frac{1}{\left|y\right|} = \frac{\left|x\right| \times 1}{\left|y\right|} = \frac{\left|x\right|}{\left|y\right|}$$

This finishes our proof.

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16. We are required to prove $\left|\frac{1}{n}\right| = \frac{1}{n}$.

Proof.

We use Proposition (I.35) $\left|xy\right| = \left|x\right|\left|y\right|$ and also $\left|\frac{1}{n}\right| = \frac{1}{n}$ because $n \in \mathbb{N}$ and the natural numbers are positive. We have

$$\left|\frac{1}{n}\right| = \left|1 \times \frac{1}{n}\right| = \left|1\right| \times \left|\frac{1}{n}\right| = 1 \times \frac{1}{n} = \frac{1}{n}.$$

This completes our proof.

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17. We can disprove $\left(n+1\right)^2 \geq 2n^2$ by giving a counter example. Consider $n = 3$, therefore

$$\left(3+1\right)^2 = 16 \leq 2\left(3^2\right) = 18.$$

(Challenge). Proof. We use proof by induction because the given result concerns natural numbers. The procedure for induction is to prove the result for $n = 3$, assume it is true for $n = k$ and then prove it for $n = k+1$. For $n = 3$ we have

$$\left(3+1\right)^2 = 16 \leq 2\left(3^2\right) = 18. \qquad \surd$$

Assume the result is true for $n = k$ that is

$$\left(k+1\right)^2 \leq 2k^2 \qquad (*)$$

Required to prove

$$\left(k+1+1\right)^2 \leq 2\left(k+1\right)^2$$

How?

By expanding the left - hand side and using the result $(k+1)^2 \le 2k^2$:

$$\begin{aligned}(k+1+1)^2 &= (k+2)^2 \\ &\le k^2+4k+4 \\ &= \underbrace{k^2+2k+1}_{=(k+1)^2}+2k+3 \\ &= (k+1)^2+2k+3\end{aligned}$$

Using (*) we have

$$\begin{aligned}(k+1+1)^2 &= (k+1)^2+2k+3 && \text{[From Above]} \\ &\le 2k^2+2k+3 && \text{[By (*)]} \\ &\le 2k^2+4k+2 = 2(k+1)^2\end{aligned}$$

Hence, we have our result for $n=k+1$. Therefore, by induction we have proven $(n+1)^2 \le 2n^2$. *(End of Challenge).*

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18. We need to evaluate $\prod_{j=1}^{5}\left(\frac{2j-1}{2j}\right)\left(\frac{2j+1}{2j}\right)$. Note that the numerator is difference of two squares and the denominator is square term. We have

$$\left(\frac{2j-1}{2j}\right)\left(\frac{2j+1}{2j}\right) = \frac{4j^2-1}{4j^2}$$

By substituting $j=1, 2, 3, 4$ and 5 we have

$$\begin{aligned}\prod_{j=1}^{5}\left(\frac{2j-1}{2j}\right)\left(\frac{2j+1}{2j}\right) &= \prod_{j=1}^{5}\left(\frac{4j^2-1}{4j^2}\right) \\ &= \left(\frac{4-1}{4}\right)\times\left(\frac{4(2^2)-1}{4(2^2)}\right)\times\left(\frac{4(3^2)-1}{4(3^2)}\right)\times\left(\frac{4(4^2)-1}{4(4^2)}\right)\times\left(\frac{4(5^2)-1}{4(5^2)}\right) \\ &= \frac{3}{4}\times\frac{15}{16}\times\frac{35}{36}\times\frac{63}{64}\times\frac{99}{100} = \frac{9\ 823\ 275}{14\ 745\ 600} = 0.666\,(3\text{sf})\end{aligned}$$

We also need to find $\left|\frac{2}{\pi}-\prod_{j=1}^{5}\left(\frac{2j-1}{2j}\right)\left(\frac{2j+1}{2j}\right)\right|$. By the above result we have

$$\left|\frac{2}{\pi}-\prod_{j=1}^{5}\left(\frac{2j-1}{2j}\right)\left(\frac{2j+1}{2j}\right)\right| = \left|\frac{2}{\pi}-0.666\right| = \left|-0.0296\right| = 0.0296\ (3\text{sf})$$

19. Here we only give some of the complete solutions as the method is identical with different numbers.

(a) We are given $x^2 - 4x + 3$ and so we have

$$x^2 - 4x + 3 = (x-2)^2 - 4 + 3 = (x-2)^2 - 1$$

(b) Similarly, for $x^2 + 7x + 1$ we have

$$\begin{aligned} x^2 + 7x + 1 &= \left(x + \frac{7}{2}\right)^2 - \left(\frac{7}{2}\right)^2 + 1 \\ &= \left(x + \frac{7}{2}\right)^2 - \frac{49}{4} + \frac{4}{4} = \left(x + \frac{7}{2}\right)^2 - \frac{45}{4} \end{aligned}$$

(e) We write the given quadratic polynomial $9 + 8x - x^2$ as

$$\begin{aligned} 9 + 8x - x^2 &= 9 - (x^2 - 8x) \\ &= 9 - \left[(x-4)^2 - 16\right] = 9 - (x-4)^2 + 16 = 25 - (x-4)^2 \end{aligned}$$

(f) We take out a factor of 3 from the given quadratic polynomial $3x^2 + 7x + 1$:

$$\begin{aligned} 3x^2 + 7x + 1 &= 3\left(x^2 + \frac{7}{3}x + \frac{1}{3}\right) \\ &= 3\left[\left(x + \frac{7}{6}\right)^2 - \left(\frac{7}{6}\right)^2 + \frac{1}{3}\right] \\ &= 3\left[\left(x + \frac{7}{6}\right)^2 - \frac{49}{36} + \frac{12}{36}\right] = 3\left[\left(x + \frac{7}{6}\right)^2 - \frac{37}{36}\right] \end{aligned}$$

Now taking the 3 in the last term on the right we have

$$3x^2 + 7x + 1 = 3\left(x + \frac{7}{6}\right)^2 - 3\left(\frac{37}{36}\right) = 3\left(x + \frac{7}{6}\right)^2 - \frac{37}{12}$$

(ii) All the hard work has been done in part (a). If we repeat the technique given in Example 44 we have:

(a) $y = x^2 - 4x + 3 = (x-2)^2 - 1 \geq 0 - 1 = -1$ gives

$$\min\{y \in \mathbb{R} : y = x^2 - 4x + 3\} = -1$$

(b) $y = x^2 + 7x + 1 = \left(x + \frac{7}{2}\right)^2 - \frac{45}{4} \geq 0 - \frac{45}{4} = -\frac{45}{4}$ we obtain

$$\min\{y \in \mathbb{R} : y = x^2 + 7x + 1\} = -\frac{45}{4}$$

(e) $y = 9 + 8x - x^2 = 25 - (x-4)^2 \leq 25 - 0 = 25$. So, we have

$$\max\{y \in \mathbb{R} : y = 9 + 8x - x^2\} = 25$$

(f) $y = 3x^2 + 7x + 1 = 3\left(x + \frac{7}{6}\right)^2 - \frac{37}{12} \geq 0 - \frac{37}{12} = -\frac{37}{12}$. Hence

$$\min\left\{y \in \mathbb{R} : y = 3x^2 + 7x + 1\right\} = -\frac{37}{12}$$

20. The complete solutions are at the following url:

Complete solutions to question 20

You need to look at solutions to question 2 and for part (l) the solution is to question 4.