Complete Solutions to Exercise **I.4**

1. We are asked to prove

$$2+4+6+\ldots+2n=\sum_{m=1}^{n}2m=n\left(n+1\right).$$

Proof.

For $n=1$ we have

$$2=1\left(1+1\right) \text{ which is true.}$$

Assume the result is true for $n=k$; that is

$$2+4+6+\ldots+2k=k\left(k+1\right) \qquad (\dagger)$$

We need to prove the result for $n=k+1$:

$$2+4+6+\ldots+2k+2\left(k+1\right)=\left(k+1\right)\left(k+2\right).$$

Expanding the left-hand side gives

$$\underbrace{2+4+6+\ldots+2k}_{=k\left(k+1\right)\text{ by }(\dagger)}+2\left(k+1\right)=k\left(k+1\right)+2\left(k+1\right)$$
$$=\left(k+1\right)\left(k+2\right) \qquad \left[\text{Factorizing}\right]$$

Thus, by mathematical induction we have our result.

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2. *Proof.* Let $P\left(n\right)$ be the given proposition: $2+5+\ldots+\left(3n-1\right)=\frac{1}{2}n\left(3n+1\right)$.

Check $P\left(1\right)$. Substituting $n=1$ gives

$$2=\frac{1}{2}\left(1\right)\left(3+1\right) \ \surd$$

Hence $P\left(1\right)$ is true. Assume the proposition is true for $n=k$:

$$2+5+8+\ldots+\left(3k-1\right)=\frac{1}{2}k\left(3k+1\right) \qquad (*)$$

Required to prove the result for $n=k+1$. We need to prove

$$2+5+8+\ldots+\left(3k-1\right)+\left(3\left(k+1\right)-1\right)=\frac{1}{2}\left(k+1\right)\left(3\left(k+1\right)+1\right)$$
$$=\frac{1}{2}\left(k+1\right)\left(3k+4\right) \qquad \left(**\right)$$

How do we prove (**)?

By examining the left-hand side and using (*).

$$2+5+\ldots+(3k-1)+(3(k+1)-1)=\underbrace{2+5+8+\ldots+(3k-1)}_{=\frac{1}{2}k(3k+1)\ \text{by (*)}}+\underbrace{(3(k+1)-1)}_{=3k+2}$$

$$=\frac{1}{2}k(3k+1)+(3k+2)$$

$$=\frac{1}{2}\left[k(3k+1)+2(3k+2)\right] \qquad \left[\text{Rewriting } (3k+2)=\frac{1}{2}2(3k+2)\right]$$

$$=\frac{1}{2}\left[3k^2+\underbrace{k+6k}_{=7k}+4\right] \qquad \left[\text{Expanding Brackets}\right]$$

$$=\frac{1}{2}\left[3k^2+7k+4\right]$$

$$=\frac{1}{2}\left[(k+1)(3k+4)\right] \qquad \left[\text{Factorizing Quadratic}\right]$$

The last line is the right-hand side of (**). Therefore, we have shown (**) and by induction we have our given proposition.

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3. We are asked to prove

$$\sum_{m=1}^{n} m^3=\frac{1}{4}n^2(n+1)^2.$$

Proof.

Check for $n=1$; $1^3=\frac{1}{4}(1)^2(1+1)^2$ which is correct.

Assume the result holds for $n=k$:

$$1^3+2^3+\cdots+k^3=\frac{1}{4}k^2(k+1)^2 \qquad (*)$$

Now we prove it for $n=k+1$:

$$1^3+2^3+\cdots+k^3+(k+1)^3=\frac{1}{4}(k+1)^2(k+2)^2.$$

Examining the left-hand side of this and using (*) we have

$$\underbrace{1^3+2^3+\cdots+k^3}_{=\frac{1}{4}k^2(k+1)^2 \text{ by } (*)}+(k+1)^3=\frac{1}{4}k^2(k+1)^2+(k+1)^3$$

$$=\frac{1}{4}k^2(k+1)^2+\frac{1}{4}4(k+1)^3 \qquad \left[\text{Writing } 1=\frac{1}{4}\times 4\right]$$

$$=\frac{1}{4}(k+1)^2\left[k^2+4(k+1)\right] \qquad \left[\text{Factorizing } \frac{1}{4}(k+1)^2\right]$$

$$=\frac{1}{4}(k+1)^2\left[k^2+4k+4\right]$$

$$=\frac{1}{4}(k+1)^2\left[k+2\right]^2 \qquad \left[\text{Because } \left[k+2\right]^2=k^2+4k+4\right]$$

Therefore we have shown $1^3+2^3+\cdots+k^3+(k+1)^3=\frac{1}{4}(k+1)^2(k+2)^2$. By mathematical induction we have our result.

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4. *Proof.* Let $P(n)$ be the given proposition: $1^3+2^3+...+n^3=(1+2+...+n)^2$.

Check $P(1)$. Substituting $n=1$ gives

$$1^3=(1)^2 \qquad \surd$$

Hence $P(1)$ is true. Assume the proposition is true for $n=k$:

$$1^3+2^3+3^3+...+k^3=(1+2+3+4+...+k)^2.$$

Required to prove the proposition for $n=k+1$:

$$1^3+2^3+3^3+...+k^3+(k+1)^3=\left(1+2+3+4+...+k+(k+1)\right)^2 \quad (\dagger)$$

Using the given hint on the left-hand side of (†) gives

$$1^3+2^3+3^3+...+k^3+(k+1)^3=\frac{1}{4}(k+1)^2(k+2)^2 \qquad (\dagger\dagger)$$

$$\left[\text{By Question 3 with } n=k+1\right]$$

How do we show this is equal to the right-hand side of (†)?

By Example 29 which is

$$1+2+3+4+...+n=\frac{1}{2}n(n+1).$$

Substituting $n=k+1$ into this we have

$$1+2+3+...+(k+1)=\frac{1}{2}(k+1)(k+2).$$

Squaring both sides gives

$$\left(1+2+3+4+\ldots+\left(k+1\right)\right)^2=\left[\frac{1}{2}\left(k+1\right)\left(k+2\right)\right]^2$$
$$=\frac{1}{4}\left(k+1\right)^2\left(k+2\right)^2$$

This the same as the right-hand side of (††). Therefore, we have shown (†) which means the result follows by induction.

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5. We are asked to prove

$$\left(1\times 2\right)+\left(2\times 3\right)+\ldots+n\left(n+1\right)=\sum_{m=1}^{n}m\left(m+1\right)=\frac{1}{3}n\left(n+1\right)\left(n+2\right).$$

Proof.

For $n=1$ we have

$$\left(1\times 2\right)=\frac{1}{3}1\left(1+1\right)\left(1+2\right)\text{ which holds.}$$

Assume the result is true for $n=k$:

$$\left(1\times 2\right)+\left(2\times 3\right)+\ldots+k\left(k+1\right)=\frac{1}{3}k\left(k+1\right)\left(k+2\right)\qquad (\dagger)$$

Required to prove the result for $n=k+1$:

$$\left(1\times 2\right)+\left(2\times 3\right)+\ldots+k\left(k+1\right)+\left(k+1\right)\left(k+2\right)=\frac{1}{3}\left(k+1\right)\left(k+2\right)\left(k+3\right)\qquad (\ddagger\ddagger)$$

We need to show that the left-hand side is equal to the right-hand side. So, considering the left-hand side

$$\underbrace{\left(1\times 2\right)+\left(2\times 3\right)+\ldots+k\left(k+1\right)}_{=\frac{1}{3}k\left(k+1\right)\left(k+2\right)\text{ by }(\dagger)}+\left(k+1\right)\left(k+2\right)=\frac{1}{3}k\left(k+1\right)\left(k+2\right)+\left(k+1\right)\left(k+2\right)$$
$$=\frac{1}{3}k\left(k+1\right)\left(k+2\right)+\frac{1}{3}3\left(k+1\right)\left(k+2\right)$$
$$=\frac{1}{3}\left(k+1\right)\left(k+2\right)\left(k+3\right)$$

We have now shown (‡‡) so our result holds by mathematical induction.

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6. We have to prove

$$\left(1\times2\times3\right)+\left(2\times3\times4\right)+\cdots+n\left(n+1\right)\left(n+2\right)=\frac{1}{4}n\left(n+1\right)\left(n+2\right)\left(n+3\right).$$

Proof.

First, we check the result for $n=1$:

$$\left(1\times2\times3\right)=\frac{1}{4}1\left(2\right)\left(3\right)\left(4\right)\text{ which is true.}$$

Secondly, we assume the result holds for $n=k$:

$$\left(1\times2\times3\right)+\left(2\times3\times4\right)+\cdots+k\left(k+1\right)\left(k+2\right)=\frac{1}{4}k\left(k+1\right)\left(k+2\right)\left(k+3\right) \qquad (*)$$

Lastly, we must prove the result holds for $n=k+1$:

$$\left(1\times2\times3\right)+\cdots+k\left(k+1\right)\left(k+2\right)+\left(k+1\right)\left(k+2\right)\left(k+3\right)=\frac{1}{4}\left(k+1\right)\left(k+2\right)\left(k+3\right)\left(k+4\right)$$

Using (*) on the left-hand side and algebra we have

$$\underbrace{\left(1\times2\times3\right)+\cdots+k\left(k+1\right)\left(k+2\right)}_{=\frac{1}{4}k(k+1)(k+2)(k+3)\text{ by }(*)}+\left(k+1\right)\left(k+2\right)\left(k+3\right)=\frac{1}{4}k\left(k+1\right)\left(k+2\right)\left(k+3\right)+\left(k+1\right)\left(k+2\right)\left(k+3\right)$$

$$=\frac{1}{4}k\left(k+1\right)\left(k+2\right)\left(k+3\right)+\frac{1}{4}4\left(k+1\right)\left(k+2\right)\left(k+3\right)$$

$$=\frac{1}{4}\left(k+1\right)\left(k+2\right)\left(k+3\right)\left(k+4\right) \quad \left[\text{Factorizing above line.}\right]$$

Hence, we have shown our result by mathematical induction.

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7. We need to prove $\sum_{m=0}^{n}2^m=2^{n+1}-1$.

Proof.

Check the result is true for $n=1$:

$$\sum_{m=0}^{1}2^m=2^0+2^1=1+2=2^{1+1}-1\text{ which is true.}$$

Assume we have the result for $n=k$:

$$\sum_{m=0}^{k}2^m=2^0+2^1+2^2+\cdots+2^k=2^{k+1}-1 \qquad (*)$$

We have to prove the result for $n=k+1$:

$$\sum_{m=0}^{k+1}2^m=2^0+2^1+2^2+\cdots+2^k+2^{k+1}=2^{k+2}-1 \qquad (**)$$

How do we show (**)?

By using (*) on the left-hand side of (**):

$$\sum_{m=0}^{k+1} 2^m = \underbrace{2^0 + 2^1 + \cdots + 2^k}_{=2^{k+1}-1 \text{ by } (*)} + 2^{k+1} = 2^{k+1} - 1 + 2^{k+1}$$
$$= 2\left(2^{k+1}\right) - 1 \quad \left[\text{because } x + x = 2x\right]$$
$$= 2^{k+2} - 1 \quad \left[\text{By using the rules of indices } a^m a^n = a^{m+n}\right]$$

We have shown (**) so by mathematical induction we have our given result.

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8. We are asked to show $\sum_{m=1}^{n}\left(2m-1\right)^3 = n^2\left(2n^2-1\right)$.

Proof.

Check the case $n = 1$:

$$\left(1\right)^3 = 1^2\left[2\left(1\right)^2 - 1\right] \text{ which holds.}$$

Assume the result is true for $n = k$:

$$\sum_{m=1}^{k}\left(2m-1\right)^3 = \left(2-1\right)^3 + \left(4-1\right)^3 + \cdots + \left(2k-1\right)^3 = k^2\left(2k^2-1\right) \qquad (\dagger)$$

Required to prove

$$\sum_{m=1}^{k+1}\left(2m-1\right)^3 = \left(2-1\right)^3 + \cdots + \left(2k-1\right)^3 + \left(2k+1\right)^3 = \left(k+1\right)^2\left[2\left(k+1\right)^2 - 1\right]$$
$$= \left(k+1\right)^2\left(2k^2+4k+1\right)$$
$$= \left(k^2+2k+1\right)\left(2k^2+4k+1\right)$$
$$\underbrace{=}_{\text{Expanding the brackets and simplifying}} 2k^4 + 8k^3 + 11k^2 + 6k + 1$$

Examine the left-hand side and apply $(\dagger)$ to the first k sum:

$$\underbrace{\left(2-1\right)^3 + \cdots + \left(2k-1\right)^3}_{=k^2\left(2k^2-1\right)} + \left(2k+1\right)^3 = k^2\left(2k^2-1\right) + \left(2k+1\right)^3$$
$$= 2k^4 - k^2 + 8k^3 + 3\left(2k\right)^2 + 3\left(2k\right) + 1$$
$$= 2k^4 + 8k^3 + 11k^2 + 6k + 1$$

Thus, we have shown the above, so our result holds by mathematical induction.

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9. *Proof.* Let $P(n)$ be the given proposition:

$$1^4+2^4+3^4+\ldots+n^4=\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}.$$

Check $P(1)$. Substituting $n=1$ gives

$$1^4=\frac{1(1+1)(2+1)(3+3-1)}{30}=\frac{1(2)(3)(5)}{30}=\frac{30}{30}=1\quad\surd$$

Hence $P(1)$ is true. Assume the proposition is true for $n=k$:

$$1^4+2^4+3^4+\ldots+k^4=\frac{k(k+1)(2k+1)(3k^2+3k-1)}{30}\qquad(*)$$

Required to prove the proposition for $n=k+1$:

$$1^4+2^4+\cdots+k^4+(k+1)^4=\frac{(k+1)((k+1)+1)(2(k+1)+1)\left(3(k+1)^2+3(k+1)-1\right)}{30}$$

$$=\frac{(k+1)(k+2)(2k+3)\left(3(k^2+2k+1)+3k+3-1\right)}{30}\quad\left[\begin{array}{l}\text{Simplifying}\\\text{and Expanding}\end{array}\right]$$

$$=\frac{(k+1)(k+2)(2k+3)(3k^2+6k+3+3k+2)}{30}$$

$$=\frac{(k+1)(k+2)(2k+3)(3k^2+9k+5)}{30}\qquad(**)$$

Expanding the left-hand side of (**) using (*) gives

$$1^4+2^4+\cdots+k^4+(k+1)^4=\underbrace{1^4+2^4+3^4+\ldots+k^4}_{=\frac{k(k+1)(2k+1)(3k^2+3k-1)}{30}\ \text{by }(*)}+(k+1)^4$$

$$=\frac{k(k+1)(2k+1)(3k^2+3k-1)}{30}+(k+1)^4$$

$$=\frac{(k+1)}{30}\left[k(2k+1)(3k^2+3k-1)+30(k+1)^3\right]$$

Expanding the square brackets gives:

$$\left[k(2k+1)(3k^2+3k-1)+30(k+1)^3\right]=(2k^2+k)(3k^2+3k-1)+30(k^3+3k^2+3k+1)$$

$$=6k^4+6k^3-2k^2+3k^3+3k^2-k+30k^3+90k^2+90k+30$$

$$=6k^4+39k^3+91k^2+89k+30$$

Left-hand side of (**) is equal to

$$\frac{(k+1)}{30}\left[k(2k+1)(3k^2+3k-1)+30(k+1)^3\right]=\frac{(k+1)}{30}\left[6k^4+39k^3+91k^2+89k+30\right]$$

Expanding the right-hand side of (**) also gives this result:

$$\frac{(k+1)(k+2)(2k+3)(3k^2+9k+5)}{30} = \frac{(k+1)}{30}\Big[\underbrace{(k+2)(2k+3)(3k^2+9k+5)}_{=6k^4+39k^3+91k^2+89k+30}\Big]$$

$$= \frac{(k+1)}{30}\left[6k^4+39k^3+91k^2+89k+30\right]$$

Hence the left-hand side is equal to the right-hand side of (**). We have shown $P(k) \Rightarrow P(k+1)$ therefore our given result follows by induction,

$$1^4+2^4+3^4+\ldots+n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}.$$

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10. *Proof.* Let $P(n)$ be the given proposition:

$$1^5+2^5+3^5+\ldots+n^5 = \frac{n^2(n+1)^2(2n^2+2n-1)}{12}.$$

Check $P(1)$. Substituting $n=1$ gives

$$1^5 = \frac{1^2(1+1)^2\left[2(1)^2+2(1)-1\right]}{12} = \frac{2^2(2+2-1)}{12} = \frac{4(3)}{12} = 1 \quad \surd$$

Hence $P(1)$ is true. Assume the proposition is true for $n=k$:

$$1^5+2^5+3^5+\ldots+k^5 = \frac{k^2(k+1)^2(2k^2+2k-1)}{12} \qquad (€)$$

Required to prove the proposition for $n=k+1$:

$$1^5+2^5+3^5+\ldots+k^5+(k+1)^5 = \frac{(k+1)^2\left((k+1)+1\right)^2\left[2(k+1)^2+2(k+1)-1\right]}{12}$$

$$= \frac{(k+1)^2(k+2)^2\left(2(k^2+2k+1)+2k+2-1\right)}{12}$$

$$= \frac{(k+1)^2(k+2)^2(2k^2+4k+2+2k+2-1)}{12}$$

$$= \frac{(k+1)^2(k+2)^2(2k^2+6k+3)}{12} \qquad (!)$$

Expanding the left-hand side of (!) using (€) gives

$$1^5 + \ddots + k^5 + (k+1)^5 = \underbrace{1^5 + 2^5 + 3^5 + \ldots + k^5}_{=\frac{k^2(k+1)^2(2k^2+2k-1)}{12}} + (k+1)^5$$

$$= \frac{k^2(k+1)^2(2k^2+2k-1)}{12} + (k+1)^5$$

$$= \frac{(k+1)^2}{12}\left[k^2(2k^2+2k-1) + 12(k+1)^3\right] \quad \left[\text{Taking Out a Common Factor of } \frac{(k+1)^2}{12}\right]$$

$$= \frac{(k+1)^2}{12}\left[2k^4 + 2k^3 - k^2 + 12(k^3 + 3k^2 + 3k + 1)\right] \quad \left[\text{Expanding Brackets}\right]$$

$$= \frac{(k+1)^2}{12}\left[2k^4 + 2k^3 - k^2 + 12k^3 + 36k^2 + 36k + 12\right]$$

$$= \frac{(k+1)^2}{12}\left[2k^4 + 14k^3 + 35k^2 + 36k + 12\right] \quad \left[\text{Collecting Like Terms}\right]$$

Expanding the right-hand side of (!) gives:

$$\frac{(k+1)^2(k+2)^2(2k^2+6k+3)}{12} = \frac{(k+1)^2}{12}\left[(k+2)^2(2k^2+6k+3)\right]$$

$$= \frac{(k+1)^2}{12}\left[(k^2+4k+4)(2k^2+6k+3)\right]$$

$$= \frac{(k+1)^2}{12}\left[2k^4 + 6k^3 + 3k^2 + 8k^3 + 24k^2 + 12k + 8k^2 + 24k + 12\right]$$

$$\left[\text{Expanding } (k^2+4k+4)(2k^2+6k+3)\right]$$

$$= \frac{(k+1)^2}{12}\left[2k^4 + 14k^3 + 35k^2 + 36k + 12\right]$$

Hence the left-hand side is equal to the Right-hand side of (!). We have shown $P(k) \Rightarrow P(k+1)$ therefore, our given result follows by induction,

$$1^5 + 2^5 + 3^5 + \ldots + n^5 = \frac{n^2(n+1)^2(2n^2+2n-1)}{12}.$$

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11. We have to prove 9 divides $10^n - 1$.

Proof.

Clearly the result holds for $n = 1$:

$$9 \mid (10-1) \text{ which is correct.}$$

Assume the result is true for $n = k$:

$$9 \mid \left(10^k - 1\right) \quad (*)$$

Required to prove that

$$9 \mid \left(10^{k+1} - 1\right) \qquad (**)$$

Examining $10^{k+1} - 1$:

$$\begin{aligned} 10^{k+1} - 1 &= 10\left(10^k\right) - 1 \\ &= (9+1)\left(10^k\right) - 1 \quad [\text{Writing } 10 = 9 + 1] \\ &= 9\left(10^k\right) + 10^k - 1 \end{aligned}$$

By (*) we have $9 \mid \left(10^k - 1\right)$ and clearly $9 \mid 10^k$ which implies that $9 \mid \left(10^{k+1} - 1\right)$. We conclude by mathematical that 9 divides $10^n - 1$.

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12. We are asked to prove $3 \mid \left(n^3 - n\right)$.

Proof.

Clearly the result holds for $n = 1$ because

$$3 \mid \left(1^3 - 1\right) \Rightarrow 3 \mid 0 \text{ and this holds as } 3 \times 0 = 0.$$

Assume the result is true for $n = k$:

$$3 \mid \left(k^3 - k\right) \qquad (\dagger)$$

We are required to prove the case for $n = k + 1$, that is

$$3 \mid \left((k+1)^3 - (k+1)\right).$$

Expanding out $(k+1)^3 - (k+1)$ gives

$$\begin{aligned} (k+1)^3 - (k+1) &= k^3 + 3k^2 + 3k + 1 - k - 1 \\ &= k^3 + 3k^2 + 2k \\ &= k^3 - k + 3k + 3k^2 \quad [\text{Writing } 2k = 3k - k] \\ &= \underbrace{k^3 - k}_{3 \mid \left(k^3 - k\right) \text{ by } (\dagger)} + 3\left(k + k^2\right) \end{aligned}$$

We know $3 \mid 3\left(k+k^2\right)$ and $3 \mid \left(k^3-k\right)$ which implies from above that $3 \mid \left[\left(k+1\right)^3-\left(k+1\right)\right]$. We conclude by mathematical induction that the given result is true.

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13. We are asked to prove $3 \mid n\left(n+1\right)\left(n+2\right)$.

Proof.

Check the case for $n=1$:

$$3 \mid 1\left(1+1\right)\left(1+2\right) \text{ which holds.}$$

Assume the result is true for $n=k$:

$$3 \mid k\left(k+1\right)\left(k+2\right) \qquad (*)$$

Required to prove the case for $n=k+1$:

$$3 \mid \left(k+1\right)\left(k+2\right)\left(k+3\right) \qquad (**)$$

Expanding out $\left(k+1\right)\left(k+2\right)\left(k+3\right)$ yields

$$\begin{aligned}\left(k+1\right)\left(k+2\right)\left(k+3\right) &= \left(k^2+3k+2\right)\left(k+3\right)\\ &= k^3+3k^2+3k^2+9k+2k+6\\ &= k^3+6k^2+11k+6\end{aligned}$$

As you may know when proving trigonometric identities one way is move in one direction and then move in the other direction and see if they meet up. If the result holds. So now we are going to expand the term $k\left(k+1\right)\left(k+2\right)$ in (*):

$$k\left(k+1\right)\left(k+2\right) = k\left(k^2+3k+2\right) = k^3+3k^2+2k\,.$$

We can express the previous derivation $k^3+6k^2+11k+6$ in terms of this last expression:

$$\begin{aligned}k^3+6k^2+11k+6 &= k^3+3k^2+2k+3k^2+9k+6 \quad \left[\begin{array}{l}\text{Writing } 6k^2=3k^2+3k^2\\ \text{and } 11k=9k+2k\end{array}\right]\\ &= k^3+3k^2+2k+3\left(k^2+3k+2\right) \quad \left[\text{Factorizing out the 3}\right]\\ &= \underbrace{k^3+3k^2+2k}_{3 \mid k^3+3k^2+2k \text{ because } k\left(k+1\right)\left(k+2\right)=k^3+3k^2+2k} + 3\left(k^2+3k+2\right)\end{aligned}$$

Therefore $3 \mid \left(k^3+6k^2+11k+6\right)$ and in the above we have already shown that

$$k^3 + 6k^2 + 11k + 6 = (k+1)(k+2)(k+3).$$

Thus, we have shown (**).

By mathematical induction we have our result.

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14. We are asked to prove $n^2 - n$ is an even number.

Proof.

We don't need to use mathematical induction this time because we have already proven this result. *Where?*

Firstly, we can rewrite $n^2 - n = n(n-1)$. *What do you notice about* $n(n-1)$*?*

They are two consecutive integers and we showed in question 3(ii) of Exercise **I.2** that two consecutive integers are even.

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15. (i) *Proof.* We first check the proposition for $n = 1$:

$$a = \frac{a(1-r)}{1-r} = a \qquad \text{[Cancelling } (1-r)\text{'s]}$$

Hence the proposition is true for $n = 1$. *What is our next step?*

Assume the proposition is true for $n = k$, that is

$$a + ar + ar^2 + ... + ar^{k-1} = \frac{a(1-r^k)}{1-r} \qquad (\$)$$

We need to prove the proposition for $n = k+1$ which is the following;

$$a + ar + ar^2 + ... + ar^{k-1} + ar^k = \frac{a(1-r^{k+1})}{1-r} \qquad (\#)$$

What do we need to prove?

Left-hand side is equa1 to the right-hand side of (#). Examining the left-hand side of (#) and using ($) we have

$$a + ar + \cdots + ar^{k-1} + ar^k = \underbrace{a + ar + ar^2 + \ldots + ar^{k-1}}_{=\frac{a\left(1-r^k\right)}{1-r} \text{ by } (\$)} + ar^k$$

$$= \frac{a\left(1-r^k\right)}{1-r} + ar^k$$

$$= \frac{a\left(1-r^k\right) + ar^k\left(1-r\right)}{1-r} \qquad \left[\text{Common Denominator}\right]$$

$$= \frac{a - ar^k + ar^k - ar^k r}{1-r} \qquad \left[\begin{array}{l}\text{Expanding Brackets}\\ \text{on Numerator}\end{array}\right]$$

$$= \frac{a - ar^{k+1}}{1-r} \qquad \left[\text{Because } -ar^k + ar^k = 0\right]$$

$$= \frac{a\left(1-r^{k+1}\right)}{1-r} \qquad [\text{Factorizing Numerator}]$$

The last line is the right-hand side of (#). Therefore, we have shown left-hand side is equa1 to the right-hand side of (#). Hence, we have our result by mathematical induction.

■

(ii) Now we are asked to prove

$$1 + r + r^2 + \ldots + r^n = \sum_{m=0}^{n} r^m = \frac{1-r^{n+1}}{1-r} \quad \left(r \neq 1\right).$$

Proof.

This is just a corollary to part (i) because in part (i) we proved:

$$\sum_{m=1}^{n} ar^{m-1} = a + ar + ar^2 + \ldots + ar^{n-1} = \frac{a\left(1-r^n\right)}{1-r}.$$

Substituting $a = 1$ and replacing n with $n-1$ into this gives our required result.

■

16. *Proof.* By applying mathematical induction, we have:

Check the result is true for $n = 1$, that is

$$\sin(x) = \frac{\cos\left(\frac{x}{2}\right) - \cos\left(\frac{2\left(1\right)+1}{2}x\right)}{2\sin\left(\frac{x}{2}\right)}$$

$$= \frac{\cos\left(\frac{x}{2}\right) - \cos\left(\frac{3}{2}x\right)}{2\sin\left(\frac{x}{2}\right)} \qquad (\dagger)$$

How do we show the right-hand side simplifies to $\sin\left(x\right)$ *?*

We need to use the trigonometric identity:

$$\cos\left(A\right) - \cos\left(B\right) = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

on the numerator of (†).

$$\cos\left(\frac{x}{2}\right) - \cos\left(\frac{3x}{2}\right) = -2\sin\left(\frac{x+3x}{4}\right)\sin\left(\frac{x-3x}{4}\right)$$

$$= -2\sin\left(x\right)\sin\left(-\frac{x}{2}\right) \qquad \left[\text{Simplifying}\right]$$

$$= -2\sin\left(x\right)\left(-\sin\left(\frac{x}{2}\right)\right) \qquad \left[\text{Because } \sin\left(-\theta\right) = -\sin\left(\theta\right)\right]$$

$$= 2\sin\left(x\right)\sin\left(\frac{x}{2}\right)$$

Substituting this into (†) gives

$$\sin\left(x\right) = \frac{2\sin\left(x\right)\sin\left(\frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)} = \sin\left(x\right) \qquad \left[\text{Cancelling } 2\sin\left(\frac{x}{2}\right)\right]$$

Hence the proposition is true for $n = 1$. Next we assume the proposition is true for $n = k$:

$$\sin(x) + \sin(2x) + \ldots + \sin(kx) = \frac{\cos\left(\frac{x}{2}\right) - \cos\left(\frac{2k+1}{2}x\right)}{2\sin\left(\frac{x}{2}\right)} \qquad (*)$$

We need to prove the proposition for $n = k+1$, that is

$$\sin(x)+\cdots+\sin(kx)+\sin\left(\left(k+1\right)x\right)=\frac{\cos\left(\frac{x}{2}\right)-\cos\left(\frac{2\left(k+1\right)+1}{2}x\right)}{2\sin\left(\frac{x}{2}\right)}$$

$$=\frac{\cos\left(\frac{x}{2}\right)-\cos\left(\left(\frac{2k+3}{2}\right)x\right)}{2\sin\left(\frac{x}{2}\right)} \qquad (**)$$

What do we need to show?

The left-hand side is equal to the right-hand side of (**). Let's examine the left-hand side first.

$$\underbrace{\sin(x)+\ldots+\sin(kx)}_{=\frac{\cos\left(\frac{x}{2}\right)-\cos\left(\frac{2k+1}{2}x\right)}{2\sin\left(\frac{x}{2}\right)} \text{ by } (*)}+\sin\left(\left(k+1\right)x\right)=\frac{\cos\left(\frac{x}{2}\right)-\cos\left(\frac{2k+1}{2}x\right)}{2\sin\left(\frac{x}{2}\right)}+\sin\left(\left(k+1\right)x\right)$$

$$=\frac{\cos\left(\frac{x}{2}\right)-\cos\left(\frac{2k+1}{2}x\right)+2\sin\left(\frac{x}{2}\right)\sin\left(\left(k+1\right)x\right)}{2\sin\left(\frac{x}{2}\right)} \quad \left[\text{Common Denominator}\right]$$

What do we do next?

We can use the following trigonometric identity on the last term of the numerator; $2\sin\left(A\right)\sin\left(B\right)=\cos\left(A-B\right)-\cos\left(A+B\right)$. We have

$$2\sin\left(\frac{x}{2}\right)\sin\left(\left(k+1\right)x\right)=\left[\cos\left(\frac{x}{2}-\left(k+1\right)x\right)-\cos\left(\frac{x}{2}+\left(k+1\right)x\right)\right]$$
$$=\left[\cos\left(\frac{x}{2}-\frac{\left(2k+2\right)x}{2}\right)-\cos\left(\frac{x}{2}+\frac{\left(2k+2\right)x}{2}\right)\right]$$
$$=\left[\cos\left(\frac{x-2kx-2x}{2}\right)-\cos\left(\frac{x+2kx+2x}{2}\right)\right]$$
$$=\left[\cos\left(\frac{-x-2kx}{2}\right)-\cos\left(\frac{3x+2kx}{2}\right)\right]$$
$$=\left[\cos\left(\frac{x+2kx}{2}\right)-\cos\left(\frac{3x+2kx}{2}\right)\right]\qquad\left[\text{Using }\cos\left(-\theta\right)=\cos\left(\theta\right)\right]$$
$$=\left[\cos\left(\frac{\left(2k+1\right)x}{2}\right)-\cos\left(\frac{\left(2k+3\right)x}{2}\right)\right]$$

Substituting this into the above we have

$$\sin(x)+\ldots+\sin\left(\left(k+1\right)x\right)=\frac{\cos\left(\frac{x}{2}\right)-\cos\left(\frac{2k+1}{2}x\right)+\left[\cos\left(\frac{\left(2k+1\right)x}{2}\right)-\cos\left(\frac{\left(2k+3\right)x}{2}\right)\right]}{2\sin\left(\frac{x}{2}\right)}$$
$$=\frac{\cos\left(\frac{x}{2}\right)-\cos\left(\frac{\left(2k+3\right)x}{2}\right)}{2\sin\left(\frac{x}{2}\right)}$$

$\left[\text{Because }-\cos\left(\frac{2k+1}{2}x\right)+\cos\left(\frac{\left(2k+1\right)x}{2}\right)=0\right]$. Hence, we have the right-hand side of (**). Therefore, we have our required result and the proposition is proved by induction.

■

17. *Proof.* We first check the proposition for $n=1$:

$$(a+b)^1=a^1+b^1=a+b\,.$$

Hence the proposition is true for $n=1$. *What is our next step?*

Assume the proposition is true for $n=k$, that is

$$\left(a+b\right)^k=a^k+ka^{k-1}b+\frac{k\left(k-1\right)}{2!}a^{k-2}b^2+\ldots+b^k\qquad(*)$$

We need to prove the proposition for $n = k+1$ which is the following;

$$\left(a+b\right)^{k+1} = a^{k+1} + \left(k+1\right)a^{k-1+1}b + \frac{\left(k+1\right)\left(\left(k+1\right)-1\right)}{2!}a^{\left(k+1\right)-2}b^2 + \ldots + b^{k+1}$$
$$= a^{k+1} + \left(k+1\right)a^k b + \frac{\left(k+1\right)k}{2!}a^{k-1}b^2 + \ldots + b^{k+1}$$

What do we need to show to prove this?

Left-hand side is equa1 to the right-hand side. *How?*

Using (*) and algebraic manipulation.

$$\left(a+b\right)^{k+1} = \left(a+b\right)^k\left(a+b\right)^1$$
$$= \left(\underbrace{a^k + ka^{k-1}b + \frac{k\left(k-1\right)}{2!}a^{k-2}b^2 + \ldots + b^k}_{\text{by (*)}}\right)\left(a+b\right)$$
$$= \underbrace{a^k a + ka^{k-1}ba + \frac{k\left(k-1\right)}{2!}a^{k-2}b^2 a + \ldots + b^k a}_{\text{Multiplying the Long Bracket by } a} + \underbrace{a^k b + ka^{k-1}bb + \frac{k\left(k-1\right)}{2!}a^{k-2}b^2 b + \ldots + b^k b}_{\text{Multiplying the Long Bracket by } b}$$
$$= a^{k+1} + ka^k b + \frac{k\left(k-1\right)}{2!}a^{k-1}b^2 + \ldots + ab^k + a^k b + ka^{k-1}b^2 + \frac{k\left(k-1\right)}{2!}a^{k-2}b^3 + \ldots + b^{k+1}$$

[Simplifying by using rules of Indices]

$$= a^{k+1} + \left(k+1\right)a^k b + \left[\frac{k\left(k-1\right)}{2!} + k\right]a^{k-1}b^2 + \ldots + b^{k+1} \quad \text{[Collecting like Terms]}$$
$$= a^{k+1} + \left(k+1\right)a^k b + \underbrace{\left[\frac{k\left(k+1\right)}{2!}\right]}_{\text{because } \frac{k\left(k-1\right)}{2!}+k=\frac{k\left(k+1\right)}{2!}} a^{k-1}b^2 + \ldots + b^{k+1}$$

Hence, we have

$$\left(a+b\right)^{k+1} = a^{k+1} + \left(k+1\right)a^k b + \frac{\left(k+1\right)k}{2!}a^{k-1}b^2 + \ldots + b^{k+1}$$

The required result. We have proven the binomial theorem for all natural numbers.

■