Complete Solutions to Exercises I.3

1. We need to show:

$$\left[\text{not } \left(P \Rightarrow Q\right)\right] \equiv \left[P \wedge \left(\text{not } Q\right)\right] \qquad \left[\text{Equivalent}\right]$$

Column 1	Column 2	Column 3	Column 4	Column 5
P	Q	$P \Rightarrow Q$	$\text{not } \left(P \Rightarrow Q\right)$	$P \wedge \left(\text{not } Q\right)$
T	T	T	F	F
T	F	F	T	T
F	T	T	F	F
F	F	T	F	F

Since the last two columns agree we have the required result.

2. (a) We are asked to show $x^2 - 3x + 2 = 0 \quad \Leftrightarrow \quad x = 1 \text{ or } x = 2$.

Proof.

$(\Leftarrow)$. Let $x = 1, x = 2$ then

$$1^2 - \left(3 \times 1\right) + 2 = 0 \text{ and } 2^2 - \left(3 \times 2\right) + 2 = 0.$$

Hence, we have $x = 1$ or $x = 2 \Rightarrow x^2 - 3x + 2 = 0$.

$(\Rightarrow)$. Now we go the other way and solve the quadratic equation $x^2 - 3x + 2 = 0$:

$$\begin{aligned} x^2 - 3x + 2 = \left(x-1\right)\left(x-2\right) = 0 \\ \Rightarrow x = 1 \text{ or } x = 2 \end{aligned}$$

This completes our proof.

(b) We are required to prove $x^2 - 10x + 21 = 0 \Leftrightarrow x = 3$ or $x = 7$.

Proof.

$(\Leftarrow)$. Let $x = 3$ or $x = 7$ then substituting into the given quadratic yields

$$3^2 - \left(10 \times 3\right) + 21 = 0 \text{ and } 7^2 - \left(10 \times 7\right) + 21 = 0.$$

$(\Rightarrow)$. Solving the quadratic

$$\begin{aligned} x^2 - 10x + 21 = \left(x-3\right)\left(x-7\right) = 0 \\ \Rightarrow \quad x = 3, \quad x = 7 \end{aligned}$$

This completes our proof.

(c) We must prove $x^2-1=0 \Leftrightarrow x=1$ or $x=-1$. It is proved very similar to parts (a) and (b). Once you are confident you can apply the $\Leftrightarrow$ in one go provided you know it works both ways:

$$\begin{aligned} x^2-1=0 &\Leftrightarrow (x-1)(x+1)=0 \\ &\Leftrightarrow x=1,\ x=-1 \end{aligned}$$

This is our required result.

(d) Very similar to the above parts, factorize the given quadratic:

$$x^2-(a+b)x+ab=0 \Leftrightarrow (x-a)(x-b)=0 \Leftrightarrow x=a,\ x=b\,.$$

(e) Rewrite

$$x^2=y^2 \Leftrightarrow x^2-y^2=0 \Leftrightarrow (x-y)(x+y)=0 \Leftrightarrow x=y,\ x=-y\,.$$

3. (a) We have to prove n is even $\Leftrightarrow$ n^2 is even.

Proof.

$(\Rightarrow)$. By Proposition (I.2) of Example 16 we have n is even $\Rightarrow$ n^2 is even.

$(\Leftarrow)$. Now we need to prove that n^2 is even $\Rightarrow$ n is even. *How?*

By using Proof by Contrapositive which means we prove that n is odd implies n^2 is odd. Let n be odd therefore $n=2m+1$ where m is an integer. Then

$$n^2=(2m+1)^2=4m^2+4m+1=2(2m^2+2m)+1\,.$$

Therefore, $n^2=2(2m^2+2m)+1$ is odd. Hence, we have n^2 is even $\Rightarrow$ n is even.

By combining both the implications we have our required result; n is even $\Leftrightarrow$ n^2 is even.

∎

(b) This time we need to prove mn is odd $\Leftrightarrow$ both m and n are odd.

Proof.

$(\Leftarrow)$. In this part we need to show that both m and n are odd $\Rightarrow$ mn is odd.

Well we have already done this in question 2(f) of the last Exercise **I.2**.

$(\Rightarrow)$. Now we need to prove mn is odd $\Rightarrow$ both m and n are odd. *How do we prove this part?*

Proof by contrapositive; that is we show if one of m or n is even then mn is even. This was proved in question 2(g) in Exercise **I.2**.

This completes our proof.

∎

(c) We are asked to prove $m+n$ is odd $\Leftrightarrow$ only m or only n is odd.

Proof.

$(\Leftarrow)$. In this part we show only m or only n is odd $\Rightarrow$ $m+n$ is odd. This means that one of the integers is even and other is odd. We have already proved that $m+n$ is odd in question 2(e) of the last Exercise **I.2**.

$(\Rightarrow)$. Now we prove $m+n$ is odd $\Rightarrow$ only m or only n is odd. *How?*

We can use proof by contradiction.

Suppose $m+n$ is odd but both m and n are also odd. This implies that we can write these as $m=2k+1$ and $n=2l+1$. Adding these integers gives

$$m+n=2k+1+2l+1=2(k+l+1).$$

This result $m+n=2(k+l+1)$ implies that $m+n$ is even. This is a contradiction. *Why?*

Because our supposition was $m+n$ is odd but now we have this sum is even.

Hence both m and n cannot be odd so only m or only n is odd.

We have shown both parts which means we completed our proof.

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(d) Now we are asked to prove mn is even $\Leftrightarrow$ at least one of m or n is even.

Proof.

$(\Leftarrow)$. In this part we show if at least one of m or n is even then mn is even.

Guess what, we have already done this in question 2(g) of Exercise **I.2**.

$(\Rightarrow)$. Now we need to prove mn is even $\Rightarrow$ at least one of m or n is even.

How do we prove this?

By contradiction.

Suppose mn is even but both m and n are odd. Then by the above part (b) of this question we have mn is odd. This is contradiction because we have mn is even and odd which is impossible. Therefore our supposition is wrong and mn is even implies at least one of of m or n is even.

This completes our proof.

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4. (a) $Q\Rightarrow P$. We cannot have $P\Rightarrow Q$ because if $a<0$ then $a^2>0$.

(b) $P\Leftrightarrow Q$. (We will prove this in section I.6 under inequalities).

(c) $Q\Rightarrow P$. We cannot have $P\Rightarrow Q$ because let $x=3.1$ so $x<4$.

(d) $P\Leftrightarrow Q$. This works both ways because $x^2-x-2=(x-2)(x+1)=0$.

(e) $P \Leftrightarrow Q$. Recall that $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ and will have two real roots if and only if $\sqrt{b^2 - 4ac} \geq 0$.

(f) $Q \Rightarrow P$. $P \not\Rightarrow Q$ for example $2 \mid (3+1)$ but $2 \nmid 3$ and $2 \nmid 1$.

(g) $P \Leftrightarrow Q$. We have proven this result in Proposition (I.7).

(h) $P \Leftrightarrow Q$ because $e^0 = 1$.

(i) $P \Leftrightarrow Q$ because $\ln(1) = 0$.

(j) $P \Leftrightarrow Q$ because a and b are positive integers so if $a < b$ then $a^n < b^n$ and also if $a^n < b^n$ then $a < b$.

(k) $P \Leftrightarrow Q$ because both x and y are positive so

$$0 < x < y \quad \Leftrightarrow \quad 0 < \frac{1}{y} < \frac{1}{x}.$$

This will be covered in section F of this chapter under inequalities.

5. Cards E and 9. Note what the statement says if there is a vowel on one side then an even number on the other side of the card. *Card E is clear but why the card 9?*

Because we have $P \Rightarrow Q$ proposition so we can check that the contrapositive:

$$\neg Q \Rightarrow \neg P.$$

6. For this question we apply the Pigeonhole Principle:

(I.8) Pigeonhole Principle: If there are $n+1$ or more objects and only n boxes then some box will contain at least two objects.

Since there are eight students and only seven days in a week so at least a couple of the students will have their birthday on the same day of the week.

7. *Proof.* Suppose there is a real number x such that it has two additive inverses call them y and z. Then $y \neq z$ because if $y = z$ then we have a unique additive inverse and there is nothing left to prove. Thus, we have

$$x + y = 0 \qquad (\dagger)$$
$$x + z = 0 \qquad (\dagger\dagger)$$

Subtracting the two equations (†) and (††) gives

$$y - z = 0 \quad \Rightarrow \quad y = z.$$

Thus, we have $y \neq z$ and $y = z$. Contradiction. Therefore, every real number has a unique additive inverse.

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8. *Proof.* Suppose $xy = 0$ and both $x \neq 0$ and $y \neq 0$. Multiply both sides of $xy = 0$ by the reciprocal of x. *What is the reciprocal of x?*

$$\frac{1}{x}.$$

Multiplying $xy = 0$ by $\frac{1}{x}$ gives

$$\frac{1}{x}(xy) = 0$$

$$y = 0 \qquad [\text{Cancelling } x'\text{s}]$$

Contradicting the supposition that $y \neq 0$. Hence the given proposition '$xy = 0 \Rightarrow x = 0$ or $y = 0$' is true.

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9. *Proof.* Suppose that n^2 is odd and n is even. We can write n as

$$n = 2m \text{ where } m \text{ is an integer.}$$

Squaring both sides of $n = 2m$ gives

$$n^2 = (2m)^2 = 4m^2 = 2(2m^2).$$

We have $n^2 = 2(\text{Integer})$ which means it is even. Hence, we have n^2 is odd and n^2 is even. This contradicts our supposition that 'n^2 is odd and n is even'. Therefore the given proposition 'n^2 is odd $\Rightarrow$ n is odd ' must be true.

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10. *Proof.* Suppose that n^3 is odd and n is even. We can write n as

$$n = 2m \text{ where } m \text{ is an integer.}$$

Cubing both sides of $n = 2m$ gives

$$n^3 = (2m)^3 = 8m^3 = 2(4m^3).$$

We have $n^3 = 2(\text{Integer})$ which means it is even. Hence, we have n^3 is odd and even which is a contradiction. Our supposition that 'n^3 is odd and n is even' must be false. Therefore, the given proposition 'n^3 is odd $\Rightarrow$ n is odd 'must be true.

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11. *Proof.* Suppose that n^3 is even and n is odd. We can write n as

$$n = 2m + 1 \text{ where } m \text{ is an integer.}$$

Cubing both sides of $n = 2m + 1$ gives

$$\begin{aligned} n^3 &= (2m+1)^3 \\ &= 8m^3 + 3(2m)^2 + 3(2m) + 1 \qquad [\text{Expanding}] \\ &= 2(4m^3) + 12m^2 + 6m + 1 = 2(4m^3 + 6m^2 + 3m) + 1 \end{aligned}$$

We have $n^3 = 2(\text{Integer}) + 1$ which means it is odd. Hence, we have n^3 is odd and n^3 is even which is a contradiction. Our supposition that 'n^3 is even and n is odd' must be false. Therefore, the given proposition 'n^3 is even $\Rightarrow$ n is even 'must be true.

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12. *Proof.* Suppose that ab is odd and a is even or b is even.
Without loss of generality assume a is even. We can write this as

$$a = 2m \text{ where } m \text{ is an integer.}$$

Therefore $ab = 2mb$ which means that ab is even. We have ab is even and ab is odd. Our supposition that 'ab is odd and a is even or b is even' leads to a contradiction therefore the given proposition 'ab is odd $\Rightarrow$ both a is odd and b is odd' is true.

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13. *Proof.* Suppose that ab is even and both a and b are odd.
We can write a and b as

$$a = 2k + 1 \text{ and } b = 2m + 1 \text{ where } k \text{ and } m \text{ are integers.}$$

Multiplying these $a = 2k + 1$ and $b = 2m + 1$ yields

$$\begin{aligned} ab &= (2k+1)(2m+1) \\ &= 4km + 2k + 2m + 1 = 2(2km + k + m) + 1 \end{aligned}$$

$ab = 2(\text{Integer}) + 1$ therefore ab is odd. Since ab is odd and ab is even is a contradiction therefore our supposition 'ab is even and both a and b are odd' is false so the given proposition 'ab is even $\Rightarrow$ a is even or b is even' is true.

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14. *Proof.* Suppose that $\sqrt{6}$ is rational. We can write $\sqrt{6}$ as

$$\frac{p}{q} = \sqrt{6}$$

where p and q have **no** factors in common apart from 1. Multiplying by q and squaring gives

$$p = \sqrt{6}q$$
$$p^2 = 6q^2 = 2\left(3q^2\right) \qquad \left[\text{Squaring}\right]$$

Since $p^2 = 2\left(\text{Integer}\right)$ therefore it is even. By lemma (I.11) we have

$$p^2 \text{ is even} \quad \Rightarrow \quad p \text{ is even}.$$

We can write $p = 2m$ where m is an integer. Substituting this, $p = 2m$, into $p^2 = 6q^2$ gives

$$4m^2 = 6q^2$$
$$2m^2 = 3q^2 \quad \left[\text{Dividing by } 2\right]$$

$3q^2$ is even therefore by the previous question we have q^2 is even because 3 is odd. By lemma (I.11)

$$q^2 \text{ is even} \Rightarrow q \text{ is even}.$$

Both p and q are even which means that they have a common factor of 2. We have a contradiction because earlier we said p and q have **no** factors in common (apart from 1) and now we have shown that they have a common factor of 2. Our initial statement that '$\sqrt{6}$ is rational' must be false therefore $\sqrt{6}$ is irrational.

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15. *Proof.* Suppose $\sqrt[3]{2}$ is rational. We can write this as

$$\frac{p}{q} = \sqrt[3]{2} \text{ where } p \text{ and } q \text{ have } \textbf{no} \text{ factors in common}.$$

Cubing both sides gives

$$\left(\frac{p}{q}\right)^3 = 2 \quad \Rightarrow \quad p^3 = 2q^3.$$

Since $p^3 = 2\left(\text{Integer}\right)$ therefore p^3 is even. By Question 11 above we have p^3 is even $\Rightarrow$ p is even. Writing $p = 2m$ where m is an integer and taking the cube of this gives:

$$p^3 = \left(2m\right)^3 = 8m^3.$$

Substituting this, $p^3 = 8m^3$, into the above $p^3 = 2q^3$ gives

$$2q^3 = 8m^3$$
$$q^3 = 4m^3 = 2\left(2m^3\right) \qquad \left[\text{Dividing by } 2\right]$$

Similarly, $q^3 = 2\left(\text{Integer}\right)$ therefore q^3 is even. Again by Question 11 we have q^3 is even $\Rightarrow$ q is even.

Both p and q are even which means that they have a common factor of 2. We have a contradiction because earlier we said p and q have **no** factors in common (apart from 1) and now we have shown that they have a common factor of 2.

Our initial statement that '$\sqrt[3]{2}$ is rational' must be false therefore $\sqrt[3]{2}$ is irrational.

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16. *Proof.* Suppose there are positive integers a and b such that

$$a^2 - b^2 = 1.$$

Since $a^2 - b^2$ is difference of two squares we can write this as

$$a^2 - b^2 = \left(a - b\right)\left(a + b\right) = 1 \qquad (*)$$

Because a and b are positive integers therefore $a + b > 1$. Dividing both sides of (*) by $a + b$ gives

$$a - b = \frac{1}{a+b} < 1 \text{ implies that } a < b + 1.$$

Combining the two results, $b < a$ and $a < b + 1$ we have

$$b < a < b + 1$$

which means that a is an integer between b and $b + 1$. Since b is an integer therefore a **cannot be an integer** because it lies between b and $b + 1$.

Contradicting our supposition that 'there are positive integers such a and b such that $a^2 - b^2 = 1$'.

Hence the given proposition 'that there are **no** positive integer solutions such that

$$a^2 - b^2 = 1\text{'}$$

must be true.

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17. (i) Suppose a is rational and b is irrational and $a + b$ is rational. We can write $a + b$ as a fraction of two integers p and $q \neq 0$;

$$a + b = \frac{p}{q}$$

$$b = \frac{p}{q} - a = \frac{p - qa}{q}$$

This means that b is rational because we have written it as a fraction of integers. Hence b is rational and irrational. Contradicts our supposition that 'a is rational and b is irrational and $a+b$ is rational'. The given proposition 'the sum of a rational and irrational number is irrational' is true.

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(ii) We are required to prove that $a+b\sqrt{n}$ is irrational.

Proof.

Since we are given that n is a non-square number so by (I.14) $\sqrt{n}$ is irrational. This implies that $b\sqrt{n}$ is irrational. *Why?*

Because if $b\sqrt{n}$ is rational then

$$b\sqrt{n}=\frac{p}{q} \quad \Rightarrow \quad \sqrt{n}=\frac{p}{bq} \text{ where } q\neq 0.$$

This implies that $\sqrt{n}$ rational. Contradicts $\sqrt{n}$ is irrational. So we have $b\sqrt{n}$ is irrational. Now applying above result (i) we have

$$a+b\sqrt{n} \text{ is irrational.}$$

This completes our proof.

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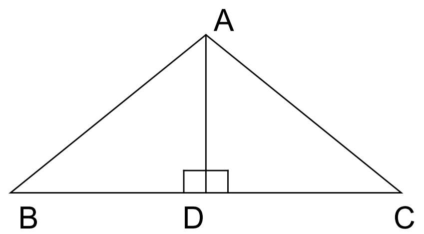
18. Suppose $\angle B=\angle C$ then $AB\neq AC$.



We have $AB\neq AC$ therefore

$$\frac{AD}{AB}\neq\frac{AD}{AC}$$

$$\sin\left(B\right)\neq\sin\left(C\right) \quad \Rightarrow \quad \angle B\neq\angle C.$$

This is a contradiction because $\angle B=\angle C$ and $\angle B\neq\angle C$. Hence the given proposition is true.

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