A Summary of Definitions and Results from the Introductory Chapter

Well Ordering Principle - WOP:

Every non-empty subset of positive integers has a *least* element.

Without Loss of Generality WLOG:

Is a simplifying assumption.

For example, say you want to prove a result concerning real numbers such as x and y. In the proof you might need to know which of the two numbers is larger, x or y. We say "Without loss of generality assume $x < y$ [x is less than y]" and then proceed with the remaining proof.

Pigeonhole Principle:

If there are $n+1$ or more objects and only n boxes then some box will contain at least two objects.

Proving

End of the proof is denoted by the symbol ■.

Pure mathematics consists of proving propositions of the type $P \Rightarrow Q$. This means if P then Q.

We also have propositions of the type $P \Leftrightarrow Q$ which means if P then Q and if Q then P. The double headed arrow goes both ways.

$P \Leftrightarrow Q$ means:

1. $P \Rightarrow Q$ and $Q \Rightarrow P$ [Implication in both directions].
2. P and Q are equivalent propositions.

Modulus Function

The **modulus** of x is denoted by $|x|$ and is defined as:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

The modulus function $|x|$ is also called the **distance** or **absolute function**.

Let x and y be real numbers then $|xy| = |x| \times |y| = |x||y|$.

Set

A **set** is a collection of objects and these objects are called **elements** or **members** of the set. The notation for a set is $\{\ \}$ and each element in the set is separated by a comma. We normally use capital letters for the set and lower-case letters as its members. A colon is used to say 'such that', for example

$$A = \left\{x : x^2 - 8x + 7 = 0\right\}$$

such that

A is the set of numbers x which satisfy $x^2 - 8x + 7 = 0$.

Inequalities of real numbers.

Let a, b and c be real numbers. We have:

(a) If $a > b$ and $c > 0$ then $ac > bc$.

(b) If $a > b$ and $c < 0$ then $ac < bc$ [The inequality changes].

Max and min of a set.

Let A be a non – empty set of numbers. Then the minimum m of the set A, denoted $\min A$, satisfies:

I. $m \in A$, that is the m is a member of the set A.

II. For every $a \in A$ we have $m \leq a$. This implies that m is less than or equal to every member of A.

Similarly, the maximum M of the set A, denoted $\max A$, satisfies:

I. $M \in A$, that is the M is a member of the set A.

II. For every $a \in A$ we have $M \geq a$. This implies that M is greater than or equal to every member of A.

For example, let $A = \left\{17, 4, 2019\right\}$ then $\min A = 4$ and $\max A = 2019$.

Sigma notation

$$\sum_{k=1}^{5}\left(k^2\right) = 1^2 + 2^2 + 3^2 + 4^2 + 5^2.$$

Product notation

$$\prod_{k=1}^{5}\left(k^2\right) = 1^2 \times 2^2 \times 3^2 \times 4^2 \times 5^2.$$

Geometric Series

A geometric series with first term equal to a and common ratio r looks like:

$$a + ar + ar^2 + \ldots + ar^{n-1}.$$

The sum of such a series is given by:

$$\sum_{m=1}^{n} ar^{m-1} = a + ar + ar^2 + \ldots + ar^{n-1} = \frac{a\left(1-r^n\right)}{1-r} \qquad [r \neq 1]$$

where a and r are real numbers.

Binomial theorem.

If a and b are real numbers, then the binomial theorem says that for all natural numbers n:

$$\left(a+b\right)^n = a^n + na^{n-1}b + \frac{n\left(n-1\right)}{2!}a^{n-2}b^2 + \frac{n\left(n-1\right)\left(n-2\right)}{3!}a^{n-3}b^3 + \ldots + b^n.$$

Rational Numbers

A rational (ratio) number is an integer or is written as a fraction of two integers, p and q, denoted by $\frac{p}{q}$ where $q \neq 0$. For example $\frac{2}{3}, -\frac{7}{101}, \frac{-5}{1} = -5, \cdots$.

A number which is *not* a rational number is called an irrational number. In other words, if we *cannot* write a number as a fraction then it is an irrational number. For example, $\pi, \sqrt{2}, e$ are all irrational numbers.

If c is a square-free number which means it is *not* a perfect square then $\sqrt{c}$ is irrational. For example

$\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{6}, \sqrt{7}, \sqrt{8}$ and $\sqrt{10}$ are *all* irrational numbers.

If a and b are integers and c is square-free then

$a + b\sqrt{c}$ is irrational number.

Identity

An **identity** is an expression which is valid for all values of the unknowns. For example, the following are identities:

$\left(a+b\right)^2 = a^2 + 2ab + b^2$ is true for all values of a and b.

$a^2 - b^2 = \left(a-b\right)\left(a+b\right)$ is true for all values of a and b.

$\sin^2\left(\theta\right) + \cos^2\left(\theta\right) = 1$ is true for all values of θ.

However $3x+1=0$ is *not* an identity because it is only true when $x=-1/3$.

Some Algebraic Identities

$$a^n - b^n = (a-b)\left(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \dots + ab^{n-2} + b^{n-1}\right).$$

$$a^{rs} - 1 = (a^r - 1)\left(a^{r(s-1)} + a^{r(s-2)} + \cdots + a^r + 1\right).$$

If n is an odd natural number then

$$\left(a^n + b^n\right) = (a+b)\left(a^{n-1} - a^{n-2}b + a^{n-3}b^2 - \cdots - ab^{n-2} + b^{n-1}\right).$$

Definition of Polynomial

An expression of the form

$$c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$$

where c's are called coefficients and are real numbers. For example

$$2x^4 - x^3 + x^2 + 5x + 7$$

is a polynomial of degree 4 because the highest index of x is 4.