Complete solutions to Exercises I.1

1. The following are propositions:

(a), (b) and (c). Only (a) is true.

2. (i) Man cannot be pregnant.

(ii) Grass is not green.

(iii) Lecturers annual salary is less than or equal to £45 000.

(iv) There are *no* integers a and b such that $\frac{a}{b} = \pi$.

(v) There are *no* integers a and b such that $\frac{a}{b} = e$.

3. If $x^2 - 9 = 0$ then $x^2 = 9$. If $x^2 = 9$ then $x = \sqrt{9}$. If $x = \sqrt{9}$ then $x = \pm 3$.

4. (i) If $x < 3$ then $x^2 < 9$.

(ii) If $x^2 < 9$ then $x < 3$.

Yes, both propositions are true.

5. (i) If ABC is an equilateral triangle then all the angles inside the triangle ABC are equal.

(ii) If all the angles inside the triangle ABC are equal then ABC is an equilateral triangle.

Both of these are true.

6. (i) If n is prime then $2^n - 1$ is prime.

(ii) If $2^n - 1$ is prime then n is prime.

Part (i) is false because 11 is prime but

$$2^{11} - 1 = 2047 = 23 \times 89.$$

Part (ii) is true.

7. The truth table is given by:

Q	P	$Q \vee P$
T	T	T
T	F	T
F	T	T
F	F	F

By comparing with the truth table for $P \vee Q$ we have

$$\left(P \vee Q\right) \equiv \left(Q \vee P\right) \qquad \text{[Equivalent].}$$

8. Truth table is

Q	P	$Q \wedge P$
T	T	T
T	F	F
F	T	F
F	F	F

By comparing with the truth table for $P \wedge Q$ we have

$$\left(P \wedge Q\right) \equiv \left(Q \wedge P\right) \quad \text{[Equivalent].}$$

9. (a)

P	$\neg P$	$\left(\neg P\right) \wedge P$
T	F	F
F	T	F

Clearly (not P) and P is going to give you false. That is

$$\left(\neg P\right) \wedge P \equiv F\,.$$

(b)

P	$\neg P$	$\left(\neg P\right) \vee P$
T	F	T
F	T	T

Clearly (not P) or P is going to give you true (T). Such a proposition is called a tautology which we discuss in the next section:

$$\left(\neg P\right) \vee P \equiv T\,.$$

(c)

P	$\neg P$	$\neg(\neg P)$
T	F	T
F	T	F

Clearly $\neg(\neg P) \equiv P$.

10. (a) $\neg(\neg(\neg P)) \equiv \neg P$

(b) $P \wedge P \equiv P$

(c) $P \vee P \equiv P$

(d) $(\neg P) \wedge (\neg P) \equiv \neg P$ [From 10(b).]

11. The truth table is

P	Q	$P \wedge Q$	$\neg(P \wedge Q)$	$\neg P$	$\neg Q$	$(\neg P) \vee (\neg Q)$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

Since the shaded columns are the same we conclude that

$$\neg(P \wedge Q) \equiv [(\neg P) \vee (\neg Q)].$$

12. Truth table is:

P	Q	R	$Q \vee R$	$P \wedge (Q \vee R)$	$P \wedge Q$	$P \wedge R$	$(P \wedge Q) \vee (P \wedge R)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T
T	F	T	T	T	F	T	T
F	T	T	T	F	F	F	F
T	F	F	F	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	F	F	F	F	F	F	F

Shaded columns agree therefore

$$P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R).$$

13. The rule is

$$\neg\neg \dots \neg P \equiv \begin{cases} P & \text{if the number of } \neg \text{ 's is even} \\ \neg P & \text{if the number of } \neg \text{ 's is odd} \end{cases}$$

Since we have 4 $\neg$'s in $\neg\neg\neg\neg P$ therefore using this rule we have

$$\neg\neg\neg\neg P \equiv P$$

which means the cup is full.

14. The truth table is

P	Q	$\neg P$	$(\neg P) \Rightarrow Q$
T	T	F	T
T	F	F	T
F	T	T	T
F	F	T	F

15. Truth table is:

P	Q	R	$\neg Q$	$P \wedge (\neg Q)$	$R \wedge (\neg R)$	$(P \wedge (\neg Q)) \Rightarrow (R \wedge (\neg R))$	$P \Rightarrow Q$	$[(P \wedge (\neg Q)) \Rightarrow (R \wedge (\neg R))] \Rightarrow (P \Rightarrow Q)$
T	T	T	F	F	F	T	T	T
T	T	F	F	F	F	T	T	T
T	F	T	T	T	F	F	F	T
F	T	T	F	F	F	T	T	T
T	F	F	T	T	F	F	F	T
F	T	F	F	F	F	T	T	T
F	F	T	T	F	F	T	T	T
F	F	F	T	F	F	T	T	T

The given compound proposition

$[(P \wedge (\neg Q)) \Rightarrow (R \wedge (\neg R))] \Rightarrow (P \Rightarrow Q)$ is always true which means it is a tautology.

Complete Solutions to Exercises I.2

1. We can construct the truth tables to show that the given propositions are tautologies.

(a)

P	$\neg P$	$(\neg P) \vee P$
T	F	T
F	T	T

Hence $(\neg P) \vee P$ is a tautology.

(b) We have

Col 1	Col 2	Col 3	Col 4	Col 5	Col 6	Col 7	Col 8	Col 9
P	Q	R	$P \Rightarrow Q$	$P \Rightarrow R$	(Col 4) $\wedge$ (Col 5)	$Q \wedge R$	$P \Rightarrow (Q \wedge R)$	(Col 6) $\Rightarrow$ (Col 8)
T	T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F	T
T	F	T	F	T	F	F	F	T
F	T	T	T	T	T	T	T	T
T	F	F	F	F	F	F	F	T
F	T	F	T	T	T	F	T	T
F	F	T	T	T	T	F	T	T
F	F	F	T	T	T	F	T	T

By looking at the right hand column we can say the following is a tautology:

$$\left[(P \Rightarrow Q) \wedge (P \Rightarrow R)\right] \Rightarrow \left[P \Rightarrow (Q \wedge R)\right].$$

(c) Similarly we have

Col 1	Col 2	Col 3	Col 4	Col 5	Col 6	Col 7	Col 8	Col 9
P	Q	R	$P \Rightarrow Q$	$R \Rightarrow Q$	(Col 4) $\wedge$ (Col 5)	$P \vee R$	$(P \vee R) \Rightarrow Q$	(Col 6) $\Rightarrow$ (Col 8)
T	T	T	T	T	T	T	T	T
T	T	F	T	T	T	T	T	T
T	F	T	F	F	F	T	F	T

F	T	T	T	T	T	T	T	T
T	F	F	F	T	F	T	F	T
F	T	F	T	T	T	F	T	T
F	F	T	T	F	F	T	F	T
F	F	F	T	T	T	F	T	T

Hence

$$\left[\left(P \Rightarrow Q\right) \wedge \left(R \Rightarrow Q\right)\right] \Rightarrow \left[\left(P \vee Q\right) \Rightarrow Q\right] \text{ is a tautology.}$$

(d) To show $\left[\left(P \Rightarrow Q\right) \wedge \left(\neg Q\right)\right] \Rightarrow \left(\neg P\right)$ is a tautology we have to construct the truth table:

P	Q	$P \Rightarrow Q$	$\neg Q$	$\left(P \Rightarrow Q\right) \wedge \left(\neg Q\right)$	$\neg P$	$\left[\left(P \Rightarrow Q\right) \wedge \left(\neg Q\right)\right] \Rightarrow \left(\neg P\right)$
T	T	T	F	F	F	T
T	F	F	T	F	F	T
F	T	T	F	F	T	T
F	F	T	T	T	T	T

The truth values in the right-hand column are *all* true therefore

$$\left[\left(P \Rightarrow Q\right) \wedge \left(\neg Q\right)\right] \Rightarrow \left(\neg P\right)$$

is a tautology.

2. (a) *Proof.* We assume m and n are even. By Definition (I.1) they can be written as

$$n = 2a \quad \text{and} \quad m = 2b$$

where a and b are integers. Consider their addition $n + m$:

$$\begin{aligned} n + m &= 2a + 2b \\ &= 2\left(a + b\right) \qquad \left[\text{Factorizing}\right] \end{aligned}$$

We have $n + m$ is of the form 2(An Integer). By applying Definition (I.1) in the $\Leftarrow$ direction we conclude that $n + m$ is even.

■

(b) *Proof.* We assume m and n are even. By Definition (I.1) they can be written as

$$n = 2a \quad \text{and} \quad m = 2b$$

where a and b are integers. Consider their subtraction $n - m$:

$$\begin{aligned} n - m &= 2a - 2b \\ &= 2\left(a - b\right) \qquad \left[\text{Factorizing}\right] \end{aligned}$$

We have $n - m$ is of the form 2(An Integer). By applying Definition (I.1) in the $\Leftarrow$ direction we conclude that $n - m$ is even.

■

(c) *Proof.* We assume m and n are odd. By Definition (I.3) they can be written as

$$n = 2a + 1 \text{ and } m = 2b + 1$$

where a and b are integers. Consider $n - m$:

$$\begin{aligned} n - m &= (2a+1) - (2b+1) \\ &= 2a - 2b \\ &= 2(a-b) \qquad [\text{Factorizing}] \end{aligned}$$

We have $n - m$ is of the form 2(An Integer). By applying Definition (I.1) in the $\Leftarrow$ direction we conclude that $n - m$ is even whenever m and n are odd.

■

(d) *Proof.* Let n be an odd number then by (I.3) there is an integer m such that $n = 2m + 1$. Consider n^2:

$$\begin{aligned} n^2 &= (2m+1)^2 \\ &= (2m+1)(2m+1) \\ &= 4m^2 + 4m + 1 = 2(2m^2 + 2m) + 1 \quad [\text{Rewriting } 4 = 2(2)] \end{aligned}$$

We have $n^2 = 2(\text{An Integer}) + 1$. By applying Definition (I.3) in the $\Leftarrow$ direction we conclude that n^2 is odd.

■

(e) *Proof.* Let n be even. Then by Definition (I.1) this can be written as

$$n = 2a \text{ where } a \text{ is an integer}.$$

Let m be odd then by (I.3) this can be written as

$$m = 2b + 1 \text{ where } b \text{ is an integer}.$$

Consider $n + m$:

$$n + m = \underbrace{2a}_{=a} + \underbrace{2b+1}_{=m} = 2(a+b) + 1 \quad [2(\text{An Integer}) + 1].$$

We have $n + m$ is 2(Integer) +1 therefore by (I.3) the number $n + m$ is odd.

■

(f) *Proof.* Let n be an odd number then by (I.3) there is an integer a such that $n = 2a + 1$. Similarly let m be an odd number then there is an integer b such that $m = 2b + 1$. Consider their product nm:

$$\begin{aligned} nm &= (2a+1)(2b+1) \\ &= 4ab + 2a + 2b + 1 \\ &= 2(2ab + a + b) + 1 \quad [2(\text{An Integer}) + 1] \end{aligned}$$

We have $nm = 2(\text{Integer}) + 1$. By applying Definition (I.3) in the $\Leftarrow$ direction we conclude that the product nm is odd.

■

(g) *Proof.* Since m is even we can write this as

$$m = 2k \text{ where } k \text{ is an integer.}$$

The product nm is given by

$$nm = n\left(2k\right) = 2kn.$$

Hence nm is a multiple of 2 therefore by Definition (I.1) we conclude that nm is even.

■

3. (i) n is odd $\Rightarrow$ $n+1$ is even.

Proof. We assume n is odd. We know n and 1 are both odd therefore by Proposition (I.4) we have $n+1$ is even.

■

(ii) For any integer n we have to show $n\left(n+1\right)$ is even because n and $n+1$ are consecutive integers.

Proof. If n is even then by the result of question 2(g) we have $n\left(n+1\right)$ is even. However, if n is odd then by the result of question 3(i) we have $n+1$ is even. Hence again by Question 2(g) we have $n\left(n+1\right)$ is even.

■

4. If n is odd then $n^3 - 1$ is even.

Proof. By the result of question 2(d) we have n is odd implies n^2 is odd. Similarly by the result of Question 2(f) we have n^2 is odd implies nn^2 is odd. Hence $nn^2 = n^3$ is odd. Since n^3 and 1 are odd therefore by the result of question 2(c) we have $n^3 - 1$ is even. This completes our proof.

■

5. (a) We need to prove $a \mid 0$.

Proof. We have $a \times 0 = 0$ therefore by Definition (I.5) we have $a \mid 0$.

(b) We need to prove $a \mid a$.

Proof. We have $a \times 1 = a$ therefore by Definition (I.5) we have $a \mid a$.

(c) We need to prove $1 \mid a$.

Proof. Since $1 \times a = a$ so by Definition (I.5) we have $1 \mid a$.

(d) Prove $a \mid a^2$.

Proof. Since $a \times a = a^2$ so by Definition (I.5) we have $a \mid a^2$.

(e) Prove $a \mid a^n$.

Proof. Since $a \times a^{n-1} = a^n$ which is $a \times \left(\text{Integer}\right) = a^n$ so by Definition (I.5) we have $a \mid a^n$.

(f) We have to prove $a \mid b$ and $a \mid c \Rightarrow a \mid \left(b+c\right)$.

Proof. We have $a \mid b$ and $a \mid c$ then

$$ax = b \text{ and } ay = c\,.$$

Therefore, adding these gives

$$b + c = ax + ay = a\left(x + y\right).$$

We have $a\left(x + y\right) = b + c$ which implies $a \mid \left(b + c\right)$ and this completes our proof.

■

(g) Need to prove: $a \mid b$ and $a \mid c \quad \Rightarrow \quad a^2 \mid bc$

Proof. From $a \mid b$ and $a \mid c$ there are integers x and y such that

$$ax = b \text{ and } ay = c\,.$$

Multiplying these together gives

$$ax\left(ay\right) = bc \quad \text{which simplifies to} \quad a^2\left(xy\right) = bc\,.$$

Since $a^2\left(\text{Integer}\right) = bc$ therefore $a^2 \mid bc$.

■

(h) Need to prove: $ac \mid bc \Rightarrow a \mid b$ where $c \neq 0$.

Proof. By using Definition (I.5) on $ac \mid bc$ we have there is an integer, x, such that

$$ac\left(x\right) = bc\,.$$

Dividing through by $c \neq 0$ gives

$$a\left(x\right) = b \text{ which implies } a \mid b\,.$$

■

(i) Prove $a \mid b$ and c $\mid d \quad \Rightarrow \quad ac \mid bd$.

Proof. From $a \mid b$ and $c \mid d$ we have integers x and y such that

$$ax = b \text{ and } cy = d\,.$$

Multiplying these together gives

$$ax\left(cy\right) = bd$$
$$ac\left(xy\right) = bd$$
$$ac\left(xy\right) = cd \quad \text{which is } ac\left(\text{Integer}\right) = bd\,.$$

By using Definition (I.5) in the direction $\Leftarrow$ we have $ac \mid bd$ which is what was required.

■

6. (a) We need to prove 'If n is odd then $8 \mid \left(n^2 - 1\right)$.'

Proof. We assume n is odd so it can be written as $n = 2m + 1$ where m is an integer. Consider $n^2 - 1$:

$$\begin{aligned} n^2 - 1 &= (2m+1)^2 - 1 \\ &= \left(\underbrace{4m^2 + 4m + 1}_{=(2m+1)^2}\right) - 1 && \text{[Expanding]} \\ &= 4m^2 + 4m = 4m(m+1) && \text{[Factorizing]} \end{aligned}$$

We know by Question 3(ii) that $m(m+1)$ is even therefore we have

$$n^2 - 1 = 4\underbrace{m(m+1)}_{\text{Even}}$$

By Definition (I.1) we can write $m(m+1) = 2k$ where k is an integer. Hence, we have

$$\begin{aligned} n^2 - 1 &= 4\underbrace{m(m+1)}_{=2k \text{ (Even)}} \\ &= 4(2k) = 8k \end{aligned}$$

Since $n^2 - 1 = 8k$ which means $8(\text{Integer}) = n^2 - 1$, therefore $8 \mid (n^2 - 1)$ and this completes our proof.

■

(b) We need to prove 'If n is odd then $32 \mid (n^2+3)(n^2+7)$'.

Proof. We assume n is odd so it can be written as $n = 2m+1$ where m is an integer. Consider the first term $n^2 + 3$ and substituting $n = 2m+1$ into this yields:

$$\begin{aligned} n^2 + 3 &= (2m+1)^2 + 3 \\ &= (4m^2 + 4m + 1) + 3 && \left[\text{Expanding } (2m+1)^2\right] \\ &= 4m^2 + 4m + 4 = 4(m^2 + m + 1) && \text{[Factorizing]} \end{aligned}$$

Similarly consider the second term $n^2 + 7$:

$$\begin{aligned} n^2 + 7 &= (2m+1)^2 + 7 \\ &= (4m^2 + 4m + 1) + 7 \\ &= 4m^2 + 4m + 8 = 4(m^2 + m + 2) \end{aligned}$$

Multiplying these together gives

$$\begin{aligned} (n^2+3)(n^2+7) &= \underbrace{4(m^2+m+1)}_{=n^2+3}\underbrace{4(m^2+m+2)}_{=n^2+7} \\ &= \underbrace{16}_{=4\times 4}(m^2+m+1)(m^2+m+2) \end{aligned}$$

Let $m^2 + m + 1 = k$ where k is an integer. Substituting this into the above we have

$$\begin{aligned} (n^2+3)(n^2+7) &= 16\underbrace{(m^2+m+1)}_{=k}\left(\underbrace{m^2+m+1}_{=k} + 1\right) \\ &= 16k(k+1) \end{aligned}$$

By question 3(ii) we have $k(k+1)$ is even therefore we can write $k(k+1)=2\ell$ where ℓ is an integer. We have

$$\begin{aligned}(n^2+3)(n^2+7) &= 16(2\ell) \qquad \left[\text{Substituting } k(k+1)=2\ell\right]\\ &= 32\ell\end{aligned}$$

We have $32(\text{Integer})=(n^2+3)(n^2+7)$. By Definition (I.5) we conclude that

$$32 \,\Big|\, (n^2+3)(n^2+7).$$

■

7. Show that if the last digit of an integer n is even then n is even.

Proof. Using the hint we have

$$\begin{aligned} n &= (a_m \times 10^m) + (a_{m-1} \times 10^{m-1}) + (a_{m-2} \times 10^{m-2}) + \ldots + (a_2 \times 10^2) + (a_1 \times 10^1) + a_0 \\ &= \left[10(a_m \times 10^{m-1}) + 10(a_{m-1} \times 10^{m-2}) + 10(a_{m-2} \times 10^{m-3}) + \ldots + 10(a_2 \times 10^1) + 10(a_1)\right] + a_0 \\ &\qquad \left[\text{Taking Out a Factor of } 10\right] \\ &= \left[\begin{aligned}&(2\times 5)(a_m \times 10^{m-1}) + (2\times 5)(a_{m-1} \times 10^{m-2}) + (2\times 5)(a_{m-2} \times 10^{m-3}) + \ldots \\ &\qquad + (2\times 5)(a_2 \times 10^1) + (2\times 5)(a_1)\end{aligned}\right] + a_0 \\ &\qquad \left[\text{Rewriting } 10 \text{ as } (2\times 5)\right] \\ &= 2\left[\begin{aligned}&5(a_m \times 10^{m-1}) + 5(a_{m-1} \times 10^{m-2}) + 5(a_{m-2} \times 10^{m-3}) + \ldots \\ &\qquad + 5(a_2 \times 10^1) + 5(a_1)\end{aligned}\right] + a_0 \end{aligned}$$

The last line says $n = 2[\text{An Integer}] + a_0$. We assume a_0 is even because the given proposition says "if the last digit of an integer n is even" and a_0 is the last digit. We can write $a_0 = 2b$. We have

$$\begin{aligned} n &= 2[\text{An Integer}] + a_0 \\ &= 2[\text{An Integer}] + 2b = 2([\text{An Integer}] + 1) \end{aligned}$$

$(\text{An Integer} + 1) = $ (Another Integer) therefore

$$n = 2(\text{Another Integer})$$

and so by (I.1) we conclude that n is even.

■

8. Show that if the last digit of an integer n is odd then n is odd.

Proof. Very similar to the proof of Question 7.

Complete Solutions to Exercises I.3

1. We need to show:

$$\left[\text{not } (P \Rightarrow Q)\right] \equiv \left[P \wedge (\text{not } Q)\right] \qquad \text{[Equivalent]}$$

Column 1	Column 2	Column 3	Column 4	Column 5
P	Q	$P \Rightarrow Q$	not $(P \Rightarrow Q)$	$P \wedge (\text{not } Q)$
T	T	T	F	F
T	F	F	T	T
F	T	T	F	F
F	F	T	F	F

Since the last two columns agree we have the required result.

2. (a) We are asked to show $x^2 - 3x + 2 = 0 \quad \Leftrightarrow \quad x = 1 \text{ or } \quad x = 2$.

Proof.

$(\Leftarrow)$. Let $x = 1, x = 2$ then

$$1^2 - (3 \times 1) + 2 = 0 \text{ and } 2^2 - (3 \times 2) + 2 = 0.$$

Hence, we have $x = 1$ or $x = 2 \Rightarrow x^2 - 3x + 2 = 0$.

$(\Rightarrow)$. Now we go the other way and solve the quadratic equation $x^2 - 3x + 2 = 0$:

$$\begin{aligned} x^2 - 3x + 2 = (x-1)(x-2) &= 0 \\ \Rightarrow x = 1 \text{ or } x &= 2 \end{aligned}$$

This completes our proof.

(b) We are required to prove $x^2 - 10x + 21 = 0 \Leftrightarrow x = 3$ or $x = 7$.

Proof.

$(\Leftarrow)$. Let $x = 3$ or $x = 7$ then substituting into the given quadratic yields

$$3^2 - (10 \times 3) + 21 = 0 \text{ and } 7^2 - (10 \times 7) + 21 = 0.$$

$(\Rightarrow)$. Solving the quadratic

$$\begin{aligned} x^2 - 10x + 21 = (x-3)(x-7) &= 0 \\ \Rightarrow x = 3, \; x &= 7 \end{aligned}$$

This completes our proof.

(c) We must prove $x^2 - 1 = 0 \Leftrightarrow x = 1$ or $x = -1$. It is proved very similar to parts (a) and (b). Once you are confident you can apply the $\Leftrightarrow$ in one go provided you know it works both ways:

$$\begin{aligned} x^2 - 1 = 0 &\Leftrightarrow (x-1)(x+1) = 0 \\ &\Leftrightarrow x = 1,\ x = -1 \end{aligned}$$

This is our required result.

(d) Very similar to the above parts, factorize the given quadratic:

$$x^2 - (a+b)x + ab = 0 \Leftrightarrow (x-a)(x-b) = 0 \Leftrightarrow x = a,\ x = b.$$

(e) Rewrite

$$x^2 = y^2 \Leftrightarrow x^2 - y^2 = 0 \Leftrightarrow (x-y)(x+y) = 0 \Leftrightarrow x = y,\ x = -y.$$

3. (a) We have to prove n is even $\Leftrightarrow$ n^2 is even.

Proof.

$(\Rightarrow)$. By Proposition (I.2) of Example 16 we have n is even $\Rightarrow$ n^2 is even.

$(\Leftarrow)$. Now we need to prove that n^2 is even $\Rightarrow$ n is even. *How?*

By using Proof by Contrapositive which means we prove that n is odd implies n^2 is odd. Let n be odd therefore $n = 2m+1$ where m is an integer. Then

$$n^2 = (2m+1)^2 = 4m^2 + 4m + 1 = 2(2m^2 + 2m) + 1.$$

Therefore, $n^2 = 2(2m^2 + 2m) + 1$ is odd. Hence, we have n^2 is even $\Rightarrow$ n is even.

By combining both the implications we have our required result; n is even $\Leftrightarrow$ n^2 is even.

■

(b) This time we need to prove mn is odd $\Leftrightarrow$ both m and n are odd.

Proof.

$(\Leftarrow)$. In this part we need to show that both m and n are odd $\Rightarrow$ mn is odd. Well we have already done this in question 2(f) of the last Exercise **I.2**.

$(\Rightarrow)$. Now we need to prove mn is odd $\Rightarrow$ both m and n are odd. *How do we prove this part?*

Proof by contrapositive; that is we show if one of m or n is even then mn is even. This was proved in question 2(g) in Exercise **I.2**.

This completes our proof.

■

(c) We are asked to prove $m+n$ is odd $\Leftrightarrow$ only m or only n is odd.

Proof.

$(\Leftarrow)$. In this part we show only m or only n is odd $\Rightarrow$ $m+n$ is odd. This means that one of the integers is even and other is odd. We have already proved that $m+n$ is odd in question 2(e) of the last Exercise **I.2**.

$(\Rightarrow)$. Now we prove $m+n$ is odd $\Rightarrow$ only m or only n is odd. *How?*

We can use proof by contradiction.

Suppose $m+n$ is odd but both m and n are also odd. This implies that we can write these as $m=2k+1$ and $n=2l+1$. Adding these integers gives

$$m+n=2k+1+2l+1=2(k+l+1).$$

This result $m+n=2(k+l+1)$ implies that $m+n$ is even. This is a contradiction. *Why?*

Because our supposition was $m+n$ is odd but now we have this sum is even. Hence both m and n cannot be odd so only m or only n is odd.

We have shown both parts which means we completed our proof.

■

(d) Now we are asked to prove mn is even $\Leftrightarrow$ at least one of m or n is even.

Proof.

$(\Leftarrow)$. In this part we show if at least one of m or n is even then mn is even.

Guess what, we have already done this in question 2(g) of Exercise **I.2**.

$(\Rightarrow)$. Now we need to prove mn is even $\Rightarrow$ at least one of m or n is even.

How do we prove this?

By contradiction.

Suppose mn is even but both m and n are odd. Then by the above part (b) of this question we have mn is odd. This is contradiction because we have mn is even and odd which is impossible. Therefore our supposition is wrong and mn is even implies at least one of of m or n is even.

This completes our proof.

■

4. (a) $Q\Rightarrow P$. We cannot have $P\Rightarrow Q$ because if $a<0$ then $a^2>0$.

(b) $P\Leftrightarrow Q$. (We will prove this in section I.6 under inequalities).

(c) $Q\Rightarrow P$. We cannot have $P\Rightarrow Q$ because let $x=3.1$ so $x<4$.

(d) $P\Leftrightarrow Q$. This works both ways because $x^2-x-2=(x-2)(x+1)=0$.

(e) $P \Leftrightarrow Q$. Recall that $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ and will have two real roots if and only if $\sqrt{b^2 - 4ac} \geq 0$.

(f) $Q \Rightarrow P$. $P \not\Rightarrow Q$ for example $2 \mid (3+1)$ but $2 \nmid 3$ and $2 \nmid 1$.

(g) $P \Leftrightarrow Q$. We have proven this result in Proposition (I.7).

(h) $P \Leftrightarrow Q$ because $e^0 = 1$.

(i) $P \Leftrightarrow Q$ because $\ln(1) = 0$.

(j) $P \Leftrightarrow Q$ because a and b are positive integers so if $a < b$ then $a^n < b^n$ and also if $a^n < b^n$ then $a < b$.

(k) $P \Leftrightarrow Q$ because both x and y are positive so

$$0 < x < y \quad \Leftrightarrow \quad 0 < \frac{1}{y} < \frac{1}{x}.$$

This will be covered in section F of this chapter under inequalities.

5. Cards E and 9. Note what the statement says if there is a vowel on one side then an even number on the other side of the card. *Card E is clear but why the card 9?*

Because we have $P \Rightarrow Q$ proposition so we can check that the contrapositive:

$$\neg Q \Rightarrow \neg P.$$

6. For this question we apply the Pigeonhole Principle:

(I.8) Pigeonhole Principle: If there are $n+1$ or more objects and only n boxes then some box will contain at least two objects.

Since there are eight students and only seven days in a week so at least a couple of the students will have their birthday on the same day of the week.

7. *Proof.* Suppose there is a real number x such that it has two additive inverses call them y and z. Then $y \neq z$ because if $y = z$ then we have a unique additive inverse and there is nothing left to prove. Thus, we have

$$x + y = 0 \qquad (\dagger)$$
$$x + z = 0 \qquad (\dagger\dagger)$$

Subtracting the two equations (†) and (††) gives

$$y - z = 0 \quad \Rightarrow \quad y = z.$$

Thus, we have $y \neq z$ and $y = z$. Contradiction. Therefore, every real number has a unique additive inverse.

■

8. *Proof.* Suppose $xy = 0$ and both $x \neq 0$ and $y \neq 0$. Multiply both sides of $xy = 0$ by the reciprocal of x. *What is the reciprocal of x?*

$$\frac{1}{x}.$$

Multiplying $xy = 0$ by $\frac{1}{x}$ gives

$$\frac{1}{x}(xy) = 0$$

$$y = 0 \qquad [\text{Cancelling } x\text{'s}]$$

Contradicting the supposition that $y \neq 0$. Hence the given proposition '$xy = 0 \Rightarrow x = 0$ or $y = 0$' is true.

■

9. *Proof.* Suppose that n^2 is odd and n is even. We can write n as

$n = 2m$ where m is an integer.

Squaring both sides of $n = 2m$ gives

$$n^2 = (2m)^2 = 4m^2 = 2(2m^2).$$

We have $n^2 = 2(\text{Integer})$ which means it is even. Hence, we have n^2 is odd and n^2 is even. This contradicts our supposition that 'n^2 is odd and n is even'. Therefore the given proposition 'n^2 is odd $\Rightarrow$ n is odd ' must be true.

■

10. *Proof.* Suppose that n^3 is odd and n is even. We can write n as

$n = 2m$ where m is an integer.

Cubing both sides of $n = 2m$ gives

$$n^3 = (2m)^3 = 8m^3 = 2(4m^3).$$

We have $n^3 = 2(\text{Integer})$ which means it is even. Hence, we have n^3 is odd and even which is a contradiction. Our supposition that 'n^3 is odd and n is even' must be false. Therefore, the given proposition 'n^3 is odd $\Rightarrow$ n is odd 'must be true.

■

11. *Proof.* Suppose that n^3 is even and n is odd. We can write n as

$$n = 2m + 1 \text{ where } m \text{ is an integer.}$$

Cubing both sides of $n = 2m + 1$ gives

$$\begin{aligned} n^3 &= (2m+1)^3 \\ &= 8m^3 + 3(2m)^2 + 3(2m) + 1 \qquad \text{[Expanding]} \\ &= 2(4m^3) + 12m^2 + 6m + 1 = 2(4m^3 + 6m^2 + 3m) + 1 \end{aligned}$$

We have $n^3 = 2(\text{Integer}) + 1$ which means it is odd. Hence, we have n^3 is odd and n^3 is even which is a contradiction. Our supposition that 'n^3 is even and n is odd' must be false. Therefore, the given proposition 'n^3 is even $\Rightarrow$ n is even 'must be true.

■

12. *Proof.* Suppose that ab is odd and a is even or b is even.
Without loss of generality assume a is even. We can write this as

$$a = 2m \text{ where } m \text{ is an integer.}$$

Therefore $ab = 2mb$ which means that ab is even. We have ab is even and ab is odd. Our supposition that 'ab is odd and a is even or b is even' leads to a contradiction therefore the given proposition 'ab is odd $\Rightarrow$ both a is odd and b is odd' is true.

■

13. *Proof.* Suppose that ab is even and both a and b are odd.
We can write a and b as

$$a = 2k + 1 \text{ and } b = 2m + 1 \text{ where } k \text{ and } m \text{ are integers.}$$

Multiplying these $a = 2k + 1$ and $b = 2m + 1$ yields

$$\begin{aligned} ab &= (2k+1)(2m+1) \\ &= 4km + 2k + 2m + 1 = 2(2km + k + m) + 1 \end{aligned}$$

$ab = 2(\text{Integer}) + 1$ therefore ab is odd. Since ab is odd and ab is even is a contradiction therefore our supposition 'ab is even and both a and b are odd' is false so the given proposition 'ab is even $\Rightarrow$ a is even or b is even' is true.

■

14. *Proof.* Suppose that $\sqrt{6}$ is rational. We can write $\sqrt{6}$ as

$$\frac{p}{q} = \sqrt{6}$$

where p and q have **no** factors in common apart from 1. Multiplying by q and squaring gives

$$p = \sqrt{6}q$$
$$p^2 = 6q^2 = 2\left(3q^2\right) \qquad \left[\text{Squaring}\right]$$

Since $p^2 = 2\left(\text{Integer}\right)$ therefore it is even. By lemma (I.11) we have

$$p^2 \text{ is even} \quad \Rightarrow \quad p \text{ is even}.$$

We can write $p = 2m$ where m is an integer. Substituting this, $p = 2m$, into $p^2 = 6q^2$ gives

$$4m^2 = 6q^2$$
$$2m^2 = 3q^2 \quad \left[\text{Dividing by } 2\right]$$

$3q^2$ is even therefore by the previous question we have q^2 is even because 3 is odd. By lemma (I.11)

$$q^2 \text{ is even} \Rightarrow q \text{ is even}.$$

Both p and q are even which means that they have a common factor of 2. We have a contradiction because earlier we said p and q have **no** factors in common (apart from 1) and now we have shown that they have a common factor of 2. Our initial statement that '$\sqrt{6}$ is rational' must be false therefore $\sqrt{6}$ is irrational.

■

15. *Proof.* Suppose $\sqrt[3]{2}$ is rational. We can write this as

$$\frac{p}{q} = \sqrt[3]{2} \text{ where } p \text{ and } q \text{ have } \textbf{no} \text{ factors in common}.$$

Cubing both sides gives

$$\left(\frac{p}{q}\right)^3 = 2 \quad \Rightarrow \quad p^3 = 2q^3.$$

Since $p^3 = 2\left(\text{Integer}\right)$ therefore p^3 is even. By Question 11 above we have p^3 is even $\Rightarrow$ p is even. Writing $p = 2m$ where m is an integer and taking the cube of this gives:

$$p^3 = \left(2m\right)^3 = 8m^3.$$

Substituting this, $p^3 = 8m^3$, into the above $p^3 = 2q^3$ gives

$$2q^3 = 8m^3$$
$$q^3 = 4m^3 = 2\left(2m^3\right) \qquad \left[\text{Dividing by } 2\right]$$

Similarly, $q^3 = 2\left(\text{Integer}\right)$ therefore q^3 is even. Again by Question 11 we have q^3 is even $\Rightarrow$ q is even.

Both p and q are even which means that they have a common factor of 2. We have a contradiction because earlier we said p and q have **no** factors in common (apart from 1) and now we have shown that they have a common factor of 2.

Our initial statement that '$\sqrt[3]{2}$ is rational' must be false therefore $\sqrt[3]{2}$ is irrational.

■

16. *Proof.* Suppose there are positive integers a and b such that

$$a^2 - b^2 = 1.$$

Since $a^2 - b^2$ is difference of two squares we can write this as

$$a^2 - b^2 = \left(a-b\right)\left(a+b\right) = 1 \qquad (*)$$

Because a and b are positive integers therefore $a+b > 1$. Dividing both sides of (*) by $a+b$ gives

$$a - b = \frac{1}{a+b} < 1 \quad \text{implies that } a < b+1.$$

Combining the two results, $b < a$ and $a < b+1$ we have

$$b < a < b+1$$

which means that a is an integer between b and $b+1$. Since b is an integer therefore a **cannot be an integer** because it lies between b and $b+1$.

Contradicting our supposition that 'there are positive integers such a and b such that $a^2 - b^2 = 1$'.

Hence the given proposition 'that there are **no** positive integer solutions such that

$$a^2 - b^2 = 1\text{'}$$

must be true.

■

17. (i) Suppose a is rational and b is irrational and $a+b$ is rational. We can write $a+b$ as a fraction of two integers p and $q \neq 0$;

$$a + b = \frac{p}{q}$$

$$b = \frac{p}{q} - a = \frac{p - qa}{q}$$

This means that b is rational because we have written it as a fraction of integers. Hence b is rational and irrational. Contradicts our supposition that 'a is rational and b is irrational and $a+b$ is rational'. The given proposition 'the sum of a rational and irrational number is irrational' is true.

■

(ii) We are required to prove that $a+b\sqrt{n}$ is irrational.

Proof.

Since we are given that n is a non-square number so by (I.14) $\sqrt{n}$ is irrational. This implies that $b\sqrt{n}$ is irrational. *Why?*

Because if $b\sqrt{n}$ is rational then

$$b\sqrt{n}=\frac{p}{q} \quad\Rightarrow\quad \sqrt{n}=\frac{p}{bq} \text{ where } q\neq 0.$$

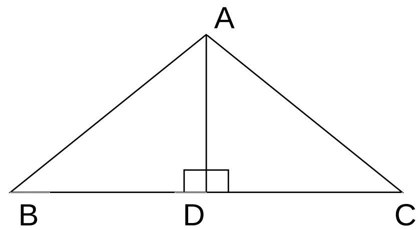
This implies that $\sqrt{n}$ rational. Contradicts $\sqrt{n}$ is irrational. So we have $b\sqrt{n}$ is irrational. Now applying above result (i) we have

$$a+b\sqrt{n} \text{ is irrational.}$$

This completes our proof.

■

18. Suppose $\angle B=\angle C$ then $AB\neq AC$.



We have $AB\neq AC$ therefore

$$\frac{AD}{AB}\neq\frac{AD}{AC}$$

$$\sin\left(B\right)\neq\sin\left(C\right) \quad\Rightarrow\quad \angle B\neq\angle C.$$

This is a contradiction because $\angle B=\angle C$ and $\angle B\neq\angle C$. Hence the given proposition is true.

■

Complete Solutions to Exercise **I.4**

1. We are asked to prove

$$2+4+6+\dots+2n=\sum_{m=1}^{n}2m=n\left(n+1\right).$$

Proof.

For $n=1$ we have

$$2=1\left(1+1\right) \text{ which is true.}$$

Assume the result is true for $n=k$; that is

$$2+4+6+\dots+2k=k\left(k+1\right) \qquad (\dagger)$$

We need to prove the result for $n=k+1$:

$$2+4+6+\dots+2k+2\left(k+1\right)=\left(k+1\right)\left(k+2\right).$$

Expanding the left-hand side gives

$$\begin{aligned}\underbrace{2+4+6+\dots+2k}_{=k\left(k+1\right)\text{ by }(\dagger)}+2\left(k+1\right)&=k\left(k+1\right)+2\left(k+1\right)\\&=\left(k+1\right)\left(k+2\right) \quad \left[\text{Factorizing}\right]\end{aligned}$$

Thus, by mathematical induction we have our result.

■

2. *Proof.* Let $P\left(n\right)$ be the given proposition: $2+5+\dots+\left(3n-1\right)=\frac{1}{2}n\left(3n+1\right)$.

Check $P\left(1\right)$. Substituting $n=1$ gives

$$2=\frac{1}{2}\left(1\right)\left(3+1\right) \ \checkmark$$

Hence $P\left(1\right)$ is true. Assume the proposition is true for $n=k$:

$$2+5+8+\dots+\left(3k-1\right)=\frac{1}{2}k\left(3k+1\right) \qquad (*)$$

Required to prove the result for $n=k+1$. We need to prove

$$\begin{aligned}2+5+8+\dots+\left(3k-1\right)+\left(3\left(k+1\right)-1\right)&=\frac{1}{2}\left(k+1\right)\left(3\left(k+1\right)+1\right)\\&=\frac{1}{2}\left(k+1\right)\left(3k+4\right) \qquad \left(**\right)\end{aligned}$$

How do we prove (**)?

By examining the left-hand side and using (*).

$$2+5+\ldots+\left(3k-1\right)+\left(3\left(k+1\right)-1\right)=\underbrace{2+5+8+\ldots+\left(3k-1\right)}_{=\frac{1}{2}k\left(3k+1\right)\text{ by (*)}}+\underbrace{\left(3\left(k+1\right)-1\right)}_{=3k+2}$$

$$=\frac{1}{2}k\left(3k+1\right)+\left(3k+2\right)$$

$$=\frac{1}{2}\left[k\left(3k+1\right)+2\left(3k+2\right)\right]\qquad\left[\text{Rewriting }\left(3k+2\right)=\frac{1}{2}2\left(3k+2\right)\right]$$

$$=\frac{1}{2}\left[3k^2+\underbrace{k+6k}_{=7k}+4\right]\qquad\left[\text{Expanding Brackets}\right]$$

$$=\frac{1}{2}\left[3k^2+7k+4\right]$$

$$=\frac{1}{2}\left[\left(k+1\right)\left(3k+4\right)\right]\qquad\left[\text{Factorizing Quadratic}\right]$$

The last line is the right-hand side of (**). Therefore, we have shown (**) and by induction we have our given proposition.

■

3. We are asked to prove

$$\sum_{m=1}^{n}m^3=\frac{1}{4}n^2\left(n+1\right)^2.$$

Proof.

Check for $n=1$; $1^3=\frac{1}{4}\left(1\right)^2\left(1+1\right)^2$ which is correct.

Assume the result holds for $n=k$:

$$1^3+2^3+\cdots+k^3=\frac{1}{4}k^2\left(k+1\right)^2\qquad(*)$$

Now we prove it for $n=k+1$:

$$1^3+2^3+\cdots+k^3+\left(k+1\right)^3=\frac{1}{4}\left(k+1\right)^2\left(k+2\right)^2.$$

Examining the left-hand side of this and using (*) we have

$$\underbrace{1^3+2^3+\cdots+k^3}_{=\frac{1}{4}k^2(k+1)^2 \text{ by } (*)}+(k+1)^3=\frac{1}{4}k^2(k+1)^2+(k+1)^3$$

$$=\frac{1}{4}k^2(k+1)^2+\frac{1}{4}4(k+1)^3 \quad \left[\text{Writing } 1=\frac{1}{4}\times 4\right]$$

$$=\frac{1}{4}(k+1)^2\left[k^2+4(k+1)\right] \quad \left[\text{Factorizing } \frac{1}{4}(k+1)^2\right]$$

$$=\frac{1}{4}(k+1)^2\left[k^2+4k+4\right]$$

$$=\frac{1}{4}(k+1)^2\left[k+2\right]^2 \quad \left[\text{Because } \left[k+2\right]^2=k^2+4k+4\right]$$

Therefore we have shown $1^3+2^3+\cdots+k^3+(k+1)^3=\frac{1}{4}(k+1)^2(k+2)^2$. By mathematical induction we have our result.

■

4. *Proof.* Let $P(n)$ be the given proposition: $1^3+2^3+...+n^3=(1+2+...+n)^2$.
Check $P(1)$. Substituting $n=1$ gives

$$1^3=(1)^2 \qquad \checkmark$$

Hence $P(1)$ is true. Assume the proposition is true for $n=k$:

$$1^3+2^3+3^3+...+k^3=(1+2+3+4+...+k)^2.$$

Required to prove the proposition for $n=k+1$:

$$1^3+2^3+3^3+...+k^3+(k+1)^3=(1+2+3+4+...+k+(k+1))^2 \quad (\dagger)$$

Using the given hint on the left-hand side of (†) gives

$$1^3+2^3+3^3+...+k^3+(k+1)^3=\frac{1}{4}(k+1)^2(k+2)^2 \qquad (\dagger\dagger)$$

$$\left[\text{By Question 3 with } n=k+1\right]$$

How do we show this is equal to the right-hand side of (†)?

By Example 29 which is

$$1+2+3+4+...+n=\frac{1}{2}n(n+1).$$

Substituting $n=k+1$ into this we have

$$1+2+3+...+(k+1)=\frac{1}{2}(k+1)(k+2).$$

Squaring both sides gives

$$\left(1+2+3+4+\ldots+\left(k+1\right)\right)^2=\left[\frac{1}{2}\left(k+1\right)\left(k+2\right)\right]^2$$
$$=\frac{1}{4}\left(k+1\right)^2\left(k+2\right)^2$$

This the same as the right-hand side of (††). Therefore, we have shown (†) which means the result follows by induction.

■

5. We are asked to prove

$$\left(1\times 2\right)+\left(2\times 3\right)+\ldots+n\left(n+1\right)=\sum_{m=1}^{n}m\left(m+1\right)=\frac{1}{3}n\left(n+1\right)\left(n+2\right).$$

Proof.

For $n=1$ we have

$$\left(1\times 2\right)=\frac{1}{3}1\left(1+1\right)\left(1+2\right) \text{ which holds.}$$

Assume the result is true for $n=k$:

$$\left(1\times 2\right)+\left(2\times 3\right)+\ldots+k\left(k+1\right)=\frac{1}{3}k\left(k+1\right)\left(k+2\right) \qquad (\dagger)$$

Required to prove the result for $n=k+1$:

$$\left(1\times 2\right)+\left(2\times 3\right)+\ldots+k\left(k+1\right)+\left(k+1\right)\left(k+2\right)=\frac{1}{3}\left(k+1\right)\left(k+2\right)\left(k+3\right) \qquad (\ddagger\ddagger)$$

We need to show that the left-hand side is equal to the right-hand side. So, considering the left-hand side

$$\underbrace{\left(1\times 2\right)+\left(2\times 3\right)+\ldots+k\left(k+1\right)}_{=\frac{1}{3}k\left(k+1\right)\left(k+2\right)\text{ by }(\dagger)}+\left(k+1\right)\left(k+2\right)=\frac{1}{3}k\left(k+1\right)\left(k+2\right)+\left(k+1\right)\left(k+2\right)$$
$$=\frac{1}{3}k\left(k+1\right)\left(k+2\right)+\frac{1}{3}3\left(k+1\right)\left(k+2\right)$$
$$=\frac{1}{3}\left(k+1\right)\left(k+2\right)\left(k+3\right)$$

We have now shown (‡‡) so our result holds by mathematical induction.

■

6. We have to prove

$$\left(1\times 2\times 3\right)+\left(2\times 3\times 4\right)+\cdots+n\left(n+1\right)\left(n+2\right)=\frac{1}{4}n\left(n+1\right)\left(n+2\right)\left(n+3\right).$$

Proof.

First, we check the result for $n=1$:

$$\left(1\times 2\times 3\right)=\frac{1}{4}1\left(2\right)\left(3\right)\left(4\right) \text{ which is true.}$$

Secondly, we assume the result holds for $n=k$:

$$\left(1\times 2\times 3\right)+\left(2\times 3\times 4\right)+\cdots+k\left(k+1\right)\left(k+2\right)=\frac{1}{4}k\left(k+1\right)\left(k+2\right)\left(k+3\right) \qquad (*)$$

Lastly, we must prove the result holds for $n=k+1$:

$$\left(1\times 2\times 3\right)+\cdots+k\left(k+1\right)\left(k+2\right)+\left(k+1\right)\left(k+2\right)\left(k+3\right)=\frac{1}{4}\left(k+1\right)\left(k+2\right)\left(k+3\right)\left(k+4\right)$$

Using (*) on the left-hand side and algebra we have

$$\underbrace{\left(1\times 2\times 3\right)+\cdots+k\left(k+1\right)\left(k+2\right)}_{=\frac{1}{4}k(k+1)(k+2)(k+3)\text{ by }(*)}+\left(k+1\right)\left(k+2\right)\left(k+3\right)=\frac{1}{4}k\left(k+1\right)\left(k+2\right)\left(k+3\right)+\left(k+1\right)\left(k+2\right)\left(k+3\right)$$

$$=\frac{1}{4}k\left(k+1\right)\left(k+2\right)\left(k+3\right)+\frac{1}{4}4\left(k+1\right)\left(k+2\right)\left(k+3\right)$$

$$=\frac{1}{4}\left(k+1\right)\left(k+2\right)\left(k+3\right)\left(k+4\right) \quad \left[\text{Factorizing above line.}\right]$$

Hence, we have shown our result by mathematical induction.

■

7. We need to prove $\sum_{m=0}^{n}2^m=2^{n+1}-1$.

Proof.

Check the result is true for $n=1$:

$$\sum_{m=0}^{1}2^m=2^0+2^1=1+2=2^{1+1}-1 \text{ which is true.}$$

Assume we have the result for $n=k$:

$$\sum_{m=0}^{k}2^m=2^0+2^1+2^2+\cdots+2^k=2^{k+1}-1 \qquad (*)$$

We have to prove the result for $n=k+1$:

$$\sum_{m=0}^{k+1}2^m=2^0+2^1+2^2+\cdots+2^k+2^{k+1}=2^{k+2}-1 \qquad (**)$$

How do we show (**)?

By using (*) on the left-hand side of (**):

$$\sum_{m=0}^{k+1} 2^m = \underbrace{2^0 + 2^1 + \cdots + 2^k}_{=2^{k+1}-1 \text{ by } (*)} + 2^{k+1} = 2^{k+1} - 1 + 2^{k+1}$$

$$= 2\left(2^{k+1}\right) - 1 \quad \left[\text{because } x + x = 2x\right]$$

$$= 2^{k+2} - 1 \quad \left[\text{By using the rules of indices } a^m a^n = a^{m+n}\right]$$

We have shown (**) so by mathematical induction we have our given result.

■

8. We are asked to show $\sum_{m=1}^{n}(2m-1)^3 = n^2\left(2n^2-1\right)$.

Proof.

Check the case $n = 1$:

$$(1)^3 = 1^2\left(2(1)^2 - 1\right) \text{ which holds.}$$

Assume the result is true for $n = k$:

$$\sum_{m=1}^{k}(2m-1)^3 = (2-1)^3 + (4-1)^3 + \cdots + (2k-1)^3 = k^2\left(2k^2-1\right) \qquad (\dagger)$$

Required to prove

$$\sum_{m=1}^{k+1}(2m-1)^3 = (2-1)^3 + \cdots + (2k-1)^3 + (2k+1)^3 = (k+1)^2\left(2(k+1)^2 - 1\right)$$

$$= (k+1)^2\left(2k^2 + 4k + 1\right)$$

$$= \left(k^2 + 2k + 1\right)\left(2k^2 + 4k + 1\right)$$

$$\underbrace{=}_{\text{Expanding the brackets and simplifying}} 2k^4 + 8k^3 + 11k^2 + 6k + 1$$

Examine the left-hand side and apply $(\dagger)$ to the first k sum:

$$\underbrace{(2-1)^3 + \cdots + (2k-1)^3}_{=k^2\left(2k^2-1\right)} + (2k+1)^3 = k^2\left(2k^2-1\right) + (2k+1)^3$$

$$= 2k^4 - k^2 + 8k^3 + 3(2k)^2 + 3(2k) + 1$$

$$= 2k^4 + 8k^3 + 11k^2 + 6k + 1$$

Thus, we have shown the above, so our result holds by mathematical induction.

■

9. *Proof.* Let $P(n)$ be the given proposition:

$$1^4+2^4+3^4+\ldots+n^4=\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}.$$

Check $P(1)$. Substituting $n=1$ gives

$$1^4=\frac{1(1+1)(2+1)(3+3-1)}{30}=\frac{1(2)(3)(5)}{30}=\frac{30}{30}=1 \quad \surd$$

Hence $P(1)$ is true. Assume the proposition is true for $n=k$:

$$1^4+2^4+3^4+\ldots+k^4=\frac{k(k+1)(2k+1)(3k^2+3k-1)}{30} \qquad (*)$$

Required to prove the proposition for $n=k+1$:

$$\begin{aligned}1^4+2^4+\cdots+k^4+(k+1)^4&=\frac{(k+1)((k+1)+1)(2(k+1)+1)\left(3(k+1)^2+3(k+1)-1\right)}{30}\\&=\frac{(k+1)(k+2)(2k+3)\left(3(k^2+2k+1)+3k+3-1\right)}{30} \quad \left[\begin{array}{l}\text{Simplifying}\\ \text{and Expanding}\end{array}\right]\\&=\frac{(k+1)(k+2)(2k+3)(3k^2+6k+3+3k+2)}{30}\\&=\frac{(k+1)(k+2)(2k+3)(3k^2+9k+5)}{30} \qquad (**)\end{aligned}$$

Expanding the left-hand side of (**) using (*) gives

$$\begin{aligned}1^4+2^4+\cdots+k^4+(k+1)^4&=\underbrace{1^4+2^4+3^4+\ldots+k^4}_{=\frac{k(k+1)(2k+1)(3k^2+3k-1)}{30} \text{ by } (*)}+(k+1)^4\\&=\frac{k(k+1)(2k+1)(3k^2+3k-1)}{30}+(k+1)^4\\&=\frac{(k+1)}{30}\left[k(2k+1)(3k^2+3k-1)+30(k+1)^3\right]\end{aligned}$$

Expanding the square brackets gives:

$$\begin{aligned}\left[k(2k+1)(3k^2+3k-1)+30(k+1)^3\right]&=(2k^2+k)(3k^2+3k-1)+30(k^3+3k^2+3k+1)\\&=6k^4+6k^3-2k^2+3k^3+3k^2-k+30k^3+90k^2+90k+30\\&=6k^4+39k^3+91k^2+89k+30\end{aligned}$$

Left-hand side of (**) is equal to

$$\frac{(k+1)}{30}\left[k(2k+1)(3k^2+3k-1)+30(k+1)^3\right]=\frac{(k+1)}{30}\left[6k^4+39k^3+91k^2+89k+30\right]$$

Expanding the right-hand side of (**) also gives this result:

$$\frac{(k+1)(k+2)(2k+3)(3k^2+9k+5)}{30} = \frac{(k+1)}{30}\Big[\underbrace{(k+2)(2k+3)(3k^2+9k+5)}_{=6k^4+39k^3+91k^2+89k+30}\Big]$$

$$= \frac{(k+1)}{30}\left[6k^4+39k^3+91k^2+89k+30\right]$$

Hence the left-hand side is equal to the right-hand side of (**). We have shown $P(k) \Rightarrow P(k+1)$ therefore our given result follows by induction,

$$1^4+2^4+3^4+\ldots+n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}.$$

■

10. *Proof.* Let $P(n)$ be the given proposition:

$$1^5+2^5+3^5+\ldots+n^5 = \frac{n^2(n+1)^2(2n^2+2n-1)}{12}.$$

Check $P(1)$. Substituting $n=1$ gives

$$1^5 = \frac{1^2(1+1)^2\left[2(1)^2+2(1)-1\right]}{12} = \frac{2^2(2+2-1)}{12} = \frac{4(3)}{12} = 1 \quad \surd$$

Hence $P(1)$ is true. Assume the proposition is true for $n=k$:

$$1^5+2^5+3^5+\ldots+k^5 = \frac{k^2(k+1)^2(2k^2+2k-1)}{12} \qquad (€)$$

Required to prove the proposition for $n=k+1$:

$$1^5+2^5+3^5+\ldots+k^5+(k+1)^5 = \frac{(k+1)^2\left((k+1)+1\right)^2\left[2(k+1)^2+2(k+1)-1\right]}{12}$$

$$= \frac{(k+1)^2(k+2)^2\left(2(k^2+2k+1)+2k+2-1\right)}{12}$$

$$= \frac{(k+1)^2(k+2)^2\left(2k^2+4k+2+2k+2-1\right)}{12}$$

$$= \frac{(k+1)^2(k+2)^2\left(2k^2+6k+3\right)}{12} \qquad (!)$$

Expanding the left-hand side of (!) using (€) gives

$$1^5+\ddots+k^5+(k+1)^5=\underbrace{1^5+2^5+3^5+\dots+k^5}_{=\frac{k^2(k+1)^2(2k^2+2k-1)}{12}}+(k+1)^5$$

$$=\frac{k^2(k+1)^2(2k^2+2k-1)}{12}+(k+1)^5$$

$$=\frac{(k+1)^2}{12}\left[k^2(2k^2+2k-1)+12(k+1)^3\right] \quad \left[\text{Taking Out a Common Factor of } \frac{(k+1)^2}{12}\right]$$

$$=\frac{(k+1)^2}{12}\left[2k^4+2k^3-k^2+12(k^3+3k^2+3k+1)\right] \quad \left[\text{Expanding Brackets}\right]$$

$$=\frac{(k+1)^2}{12}\left[2k^4+2k^3-k^2+12k^3+36k^2+36k+12\right]$$

$$=\frac{(k+1)^2}{12}\left[2k^4+14k^3+35k^2+36k+12\right] \quad \left[\text{Collecting Like Terms}\right]$$

Expanding the right-hand side of (!) gives:

$$\frac{(k+1)^2(k+2)^2(2k^2+6k+3)}{12}=\frac{(k+1)^2}{12}\left[(k+2)^2(2k^2+6k+3)\right]$$

$$=\frac{(k+1)^2}{12}\left[(k^2+4k+4)(2k^2+6k+3)\right]$$

$$=\frac{(k+1)^2}{12}\left[2k^4+6k^3+3k^2+8k^3+24k^2+12k+8k^2+24k+12\right]$$

$$\left[\text{Expanding } (k^2+4k+4)(2k^2+6k+3)\right]$$

$$=\frac{(k+1)^2}{12}\left[2k^4+14k^3+35k^2+36k+12\right]$$

Hence the left-hand side is equal to the Right-hand side of (!). We have shown $P(k)\Rightarrow P(k+1)$ therefore, our given result follows by induction,

$$1^5+2^5+3^5+\dots+n^5=\frac{n^2(n+1)^2(2n^2+2n-1)}{12}.$$

■

11. We have to prove 9 divides 10^n-1.

Proof.

Clearly the result holds for $n=1$:

$$9\,\big|\,(10-1) \quad \text{which is correct.}$$

Assume the result is true for $n = k$:

$$9 \mid \left(10^k - 1\right) \quad (*)$$

Required to prove that

$$9 \mid \left(10^{k+1} - 1\right) \qquad (**)$$

Examining $10^{k+1} - 1$:

$$\begin{aligned} 10^{k+1} - 1 &= 10\left(10^k\right) - 1 \\ &= (9+1)\left(10^k\right) - 1 \quad [\text{Writing } 10 = 9+1] \\ &= 9\left(10^k\right) + 10^k - 1 \end{aligned}$$

By (*) we have $9 \mid \left(10^k - 1\right)$ and clearly $9 \mid 10^k$ which implies that $9 \mid \left(10^{k+1} - 1\right)$. We conclude by mathematical that 9 divides $10^n - 1$.

■

12. We are asked to prove $3 \mid \left(n^3 - n\right)$.

Proof.

Clearly the result holds for $n = 1$ because

$$3 \mid \left(1^3 - 1\right) \Rightarrow 3 \mid 0 \text{ and this holds as } 3 \times 0 = 0.$$

Assume the result is true for $n = k$:

$$3 \mid \left(k^3 - k\right) \qquad (\dagger)$$

We are required to prove the case for $n = k + 1$, that is

$$3 \mid \left((k+1)^3 - (k+1)\right).$$

Expanding out $(k+1)^3 - (k+1)$ gives

$$\begin{aligned} (k+1)^3 - (k+1) &= k^3 + 3k^2 + 3k + 1 - k - 1 \\ &= k^3 + 3k^2 + 2k \\ &= k^3 - k + 3k + 3k^2 \quad [\text{Writing } 2k = 3k - k] \\ &= \underbrace{k^3 - k}_{3 \mid \left(k^3 - k\right) \text{ by } (\dagger)} + 3\left(k + k^2\right) \end{aligned}$$

We know $3 \mid 3\left(k+k^2\right)$ and $3 \mid \left(k^3-k\right)$ which implies from above that $3 \mid \left[\left(k+1\right)^3-\left(k+1\right)\right]$. We conclude by mathematical induction that the given result is true.

■

13. We are asked to prove $3 \mid n\left(n+1\right)\left(n+2\right)$.

Proof.

Check the case for $n=1$:

$$3 \mid 1\left(1+1\right)\left(1+2\right) \text{ which holds.}$$

Assume the result is true for $n=k$:

$$3 \mid k\left(k+1\right)\left(k+2\right) \qquad (*)$$

Required to prove the case for $n=k+1$:

$$3 \mid \left(k+1\right)\left(k+2\right)\left(k+3\right) \qquad (**)$$

Expanding out $\left(k+1\right)\left(k+2\right)\left(k+3\right)$ yields

$$\begin{aligned}\left(k+1\right)\left(k+2\right)\left(k+3\right) &= \left(k^2+3k+2\right)\left(k+3\right)\\ &= k^3+3k^2+3k^2+9k+2k+6\\ &= k^3+6k^2+11k+6\end{aligned}$$

As you may know when proving trigonometric identities one way is move in one direction and then move in the other direction and see if they meet up. If the result holds. So now we are going to expand the term $k\left(k+1\right)\left(k+2\right)$ in (*):

$$k\left(k+1\right)\left(k+2\right)=k\left(k^2+3k+2\right)=k^3+3k^2+2k.$$

We can express the previous derivation $k^3+6k^2+11k+6$ in terms of this last expression:

$$\begin{aligned}k^3+6k^2+11k+6 &= k^3+3k^2+2k+3k^2+9k+6 && \left[\begin{array}{l}\text{Writing } 6k^2=3k^2+3k^2\\ \text{and } 11k=9k+2k\end{array}\right]\\ &= k^3+3k^2+2k+3\left(k^2+3k+2\right) && \left[\text{Factorizing out the } 3\right]\\ &= \underbrace{k^3+3k^2+2k}_{3 \mid k^3+3k^2+2k \text{ because } k\left(k+1\right)\left(k+2\right)=k^3+3k^2+2k}+3\left(k^2+3k+2\right)\end{aligned}$$

Therefore $3 \mid \left(k^3+6k^2+11k+6\right)$ and in the above we have already shown that

$$k^3 + 6k^2 + 11k + 6 = (k+1)(k+2)(k+3).$$

Thus, we have shown (**).

By mathematical induction we have our result.

■

14. We are asked to prove $n^2 - n$ is an even number.

Proof.

We don't need to use mathematical induction this time because we have already proven this result. *Where?*

Firstly, we can rewrite $n^2 - n = n(n-1)$. *What do you notice about* $n(n-1)$*?*

They are two consecutive integers and we showed in question 3(ii) of Exercise **I.2** that two consecutive integers are even.

■

15. (i) *Proof.* We first check the proposition for $n = 1$:

$$a = \frac{a(1-r)}{1-r} = a \qquad \text{[Cancelling } (1-r)\text{'s]}$$

Hence the proposition is true for $n = 1$. *What is our next step?*

Assume the proposition is true for $n = k$, that is

$$a + ar + ar^2 + \dots + ar^{k-1} = \frac{a(1-r^k)}{1-r} \qquad (\$)$$

We need to prove the proposition for $n = k+1$ which is the following;

$$a + ar + ar^2 + \dots + ar^{k-1} + ar^k = \frac{a(1-r^{k+1})}{1-r} \qquad (\#)$$

What do we need to prove?

Left-hand side is equa1 to the right-hand side of (#). Examining the left-hand side of (#) and using ($) we have

$$a+ar+\cdots+ar^{k-1}+ar^{k}=\underbrace{a+ar+ar^{2}+...+ar^{k-1}}_{=\frac{a\left(1-r^{k}\right)}{1-r}\text{ by (\$)}}+ar^{k}$$

$$=\frac{a\left(1-r^{k}\right)}{1-r}+ar^{k}$$

$$=\frac{a\left(1-r^{k}\right)+ar^{k}\left(1-r\right)}{1-r} \qquad \left[\text{Common Denominator}\right]$$

$$=\frac{a-ar^{k}+ar^{k}-ar^{k}r}{1-r} \qquad \left[\begin{array}{l}\text{Expanding Brackets}\\ \text{on Numerator}\end{array}\right]$$

$$=\frac{a-ar^{k+1}}{1-r} \qquad \left[\text{Because } -ar^{k}+ar^{k}=0\right]$$

$$=\frac{a\left(1-r^{k+1}\right)}{1-r} \qquad [\text{Factorizing Numerator}]$$

The last line is the right-hand side of (#). Therefore, we have shown left-hand side is equa1 to the right-hand side of (#). Hence, we have our result by mathematical induction.

■

(ii) Now we are asked to prove

$$1+r+r^{2}+...+r^{n}=\sum_{m=0}^{n}r^{m}=\frac{1-r^{n+1}}{1-r} \quad \left(r\neq 1\right).$$

Proof.

This is just a corollary to part (i) because in part (i) we proved:

$$\sum_{m=1}^{n}ar^{m-1}=a+ar+ar^{2}+...+ar^{n-1}=\frac{a\left(1-r^{n}\right)}{1-r}.$$

Substituting $a=1$ and replacing n with $n-1$ into this gives our required result.

■

16. *Proof.* By applying mathematical induction, we have:

Check the result is true for $n=1$, that is

$$\sin(x) = \frac{\cos\left(\frac{x}{2}\right) - \cos\left(\frac{2\left(1\right)+1}{2}x\right)}{2\sin\left(\frac{x}{2}\right)}$$

$$= \frac{\cos\left(\frac{x}{2}\right) - \cos\left(\frac{3}{2}x\right)}{2\sin\left(\frac{x}{2}\right)} \qquad (\dagger)$$

How do we show the right-hand side simplifies to $\sin\left(x\right)$ *?*

We need to use the trigonometric identity:

$$\cos\left(A\right) - \cos\left(B\right) = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

on the numerator of (†).

$$\cos\left(\frac{x}{2}\right) - \cos\left(\frac{3x}{2}\right) = -2\sin\left(\frac{x+3x}{4}\right)\sin\left(\frac{x-3x}{4}\right)$$

$$= -2\sin\left(x\right)\sin\left(-\frac{x}{2}\right) \qquad \left[\text{Simplifying}\right]$$

$$= -2\sin\left(x\right)\left(-\sin\left(\frac{x}{2}\right)\right) \qquad \left[\text{Because } \sin\left(-\theta\right) = -\sin\left(\theta\right)\right]$$

$$= 2\sin\left(x\right)\sin\left(\frac{x}{2}\right)$$

Substituting this into (†) gives

$$\sin\left(x\right) = \frac{2\sin\left(x\right)\sin\left(\frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)} = \sin\left(x\right) \qquad \left[\text{Cancelling } 2\sin\left(\frac{x}{2}\right)\right]$$

Hence the proposition is true for $n = 1$. Next we assume the proposition is true for $n = k$:

$$\sin(x) + \sin(2x) + \ldots + \sin(kx) = \frac{\cos\left(\frac{x}{2}\right) - \cos\left(\frac{2k+1}{2}x\right)}{2\sin\left(\frac{x}{2}\right)} \qquad (*)$$

We need to prove the proposition for $n = k + 1$, that is

$$\sin(x)+\cdots+\sin(kx)+\sin\left(\left(k+1\right)x\right)=\frac{\cos\left(\frac{x}{2}\right)-\cos\left(\frac{2\left(k+1\right)+1}{2}x\right)}{2\sin\left(\frac{x}{2}\right)}$$

$$=\frac{\cos\left(\frac{x}{2}\right)-\cos\left(\left(\frac{2k+3}{2}\right)x\right)}{2\sin\left(\frac{x}{2}\right)} \qquad (**)$$

What do we need to show?

The left-hand side is equal to the right-hand side of (**). Let's examine the left-hand side first.

$$\underbrace{\sin(x)+\ldots+\sin(kx)}_{=\frac{\cos\left(\frac{x}{2}\right)-\cos\left(\frac{2k+1}{2}x\right)}{2\sin\left(\frac{x}{2}\right)} \text{ by } (*)}+\sin\left(\left(k+1\right)x\right)=\frac{\cos\left(\frac{x}{2}\right)-\cos\left(\frac{2k+1}{2}x\right)}{2\sin\left(\frac{x}{2}\right)}+\sin\left(\left(k+1\right)x\right)$$

$$=\frac{\cos\left(\frac{x}{2}\right)-\cos\left(\frac{2k+1}{2}x\right)+2\sin\left(\frac{x}{2}\right)\sin\left(\left(k+1\right)x\right)}{2\sin\left(\frac{x}{2}\right)} \quad \left[\text{Common Denominator}\right]$$

What do we do next?

We can use the following trigonometric identity on the last term of the numerator; $2\sin\left(A\right)\sin\left(B\right)=\cos\left(A-B\right)-\cos\left(A+B\right)$. We have

$$\begin{aligned}2\sin\left(\frac{x}{2}\right)\sin\left(\left(k+1\right)x\right) &= \left[\cos\left(\frac{x}{2}-\left(k+1\right)x\right)-\cos\left(\frac{x}{2}+\left(k+1\right)x\right)\right]\\ &= \left[\cos\left(\frac{x}{2}-\frac{\left(2k+2\right)x}{2}\right)-\cos\left(\frac{x}{2}+\frac{\left(2k+2\right)x}{2}\right)\right]\\ &= \left[\cos\left(\frac{x-2kx-2x}{2}\right)-\cos\left(\frac{x+2kx+2x}{2}\right)\right]\\ &= \left[\cos\left(\frac{-x-2kx}{2}\right)-\cos\left(\frac{3x+2kx}{2}\right)\right]\\ &= \left[\cos\left(\frac{x+2kx}{2}\right)-\cos\left(\frac{3x+2kx}{2}\right)\right] \quad \left[\text{Using } \cos\left(-\theta\right)=\cos\left(\theta\right)\right]\\ &= \left[\cos\left(\frac{\left(2k+1\right)x}{2}\right)-\cos\left(\frac{\left(2k+3\right)x}{2}\right)\right]\end{aligned}$$

Substituting this into the above we have

$$\begin{aligned}\sin(x)+\ldots+\sin\left(\left(k+1\right)x\right) &= \frac{\cos\left(\frac{x}{2}\right)-\cos\left(\frac{2k+1}{2}x\right)+\left[\cos\left(\frac{\left(2k+1\right)x}{2}\right)-\cos\left(\frac{\left(2k+3\right)x}{2}\right)\right]}{2\sin\left(\frac{x}{2}\right)}\\ &= \frac{\cos\left(\frac{x}{2}\right)-\cos\left(\frac{\left(2k+3\right)x}{2}\right)}{2\sin\left(\frac{x}{2}\right)}\end{aligned}$$

$\left[\text{Because } -\cos\left(\frac{2k+1}{2}x\right)+\cos\left(\frac{\left(2k+1\right)x}{2}\right)=0\right]$. Hence, we have the right-hand side of (**). Therefore, we have our required result and the proposition is proved by induction.

■

17. *Proof.* We first check the proposition for $n=1$:

$$(a+b)^1 = a^1 + b^1 = a+b\,.$$

Hence the proposition is true for $n=1$. *What is our next step?*

Assume the proposition is true for $n=k$, that is

$$\left(a+b\right)^k = a^k + ka^{k-1}b + \frac{k\left(k-1\right)}{2!}a^{k-2}b^2 + \ldots + b^k \qquad (*)$$

We need to prove the proposition for $n = k+1$ which is the following;

$$\left(a+b\right)^{k+1} = a^{k+1} + \left(k+1\right)a^{k-1+1}b + \frac{\left(k+1\right)\left(\left(k+1\right)-1\right)}{2!}a^{\left(k+1\right)-2}b^2 + \ldots + b^{k+1}$$
$$= a^{k+1} + \left(k+1\right)a^k b + \frac{\left(k+1\right)k}{2!}a^{k-1}b^2 + \ldots + b^{k+1}$$

What do we need to show to prove this?

Left-hand side is equa1 to the right-hand side. *How?*

Using (*) and algebraic manipulation.

$$\left(a+b\right)^{k+1} = \left(a+b\right)^k\left(a+b\right)^1$$
$$= \left(\underbrace{a^k + ka^{k-1}b + \frac{k\left(k-1\right)}{2!}a^{k-2}b^2 + \ldots + b^k}_{\text{by (*)}}\right)\left(a+b\right)$$
$$= \underbrace{a^k a + ka^{k-1}ba + \frac{k\left(k-1\right)}{2!}a^{k-2}b^2 a + \ldots + b^k a +}_{\text{Multiplying the Long Bracket by } a}$$
$$\underbrace{a^k b + ka^{k-1}bb + \frac{k\left(k-1\right)}{2!}a^{k-2}b^2 b + \ldots + b^k b}_{\text{Multiplying the Long Bracket by } b}$$
$$= a^{k+1} + ka^k b + \frac{k\left(k-1\right)}{2!}a^{k-1}b^2 + \ldots + ab^k +$$
$$a^k b + ka^{k-1}b^2 + \frac{k\left(k-1\right)}{2!}a^{k-2}b^3 + \ldots + b^{k+1}$$

[Simplifying by using rules of Indices]

$$= a^{k+1} + \left(k+1\right)a^k b + \left[\frac{k\left(k-1\right)}{2!} + k\right]a^{k-1}b^2 + \ldots + b^{k+1} \qquad \left[\begin{matrix}\text{Collecting}\\ \text{like Terms}\end{matrix}\right]$$
$$= a^{k+1} + \left(k+1\right)a^k b + \underbrace{\left[\frac{k\left(k+1\right)}{2!}\right]}_{\text{because } \frac{k\left(k-1\right)}{2!}+k=\frac{k\left(k+1\right)}{2!}} a^{k-1}b^2 + \ldots + b^{k+1}$$

Hence, we have

$$\left(a+b\right)^{k+1} = a^{k+1} + \left(k+1\right)a^k b + \frac{\left(k+1\right)k}{2!}a^{k-1}b^2 + \ldots + b^{k+1}$$

The required result. We have proven the binomial theorem for all natural numbers.

■

Complete Solutions to Exercises I.5

1. In each case we solve the given equation and we obtain the following results:

(a) $A = \left\{-\frac{1}{2}\right\}$ (b) $B = \{1,\ 2\}$ (c) $C = \{1\}$

(d) $D = \{-1,\ 5\}$ (e) $E = \{-3,\ 3\}$

(f) $F = \{2,\ 3,\ 5,\ 7\}$ (these are the prime numbers less than 10.)

2. This time be careful because you need to look at the universal set.

(a) The solution to the give quadratic equation

$$(x-1)(x+3) = 0 \ \Rightarrow\ x = 1,\ x = -3\,.$$

However our universal set is that natural numbers $\mathbb{N}$ so the only member of the given set A is 1, that is $A = \{1\}$.

(b) This time solving $2x + 1 = 0 \ \Rightarrow\ x = -\frac{1}{2}$. But we are given

$$B = \{x \in \mathbb{N} : 2x + 1 = 0\}\,.$$

Clearly $-\frac{1}{2}$ is *not* a natural number so $B = \varnothing$.

(c) We are given the set $C = \{x \in \mathbb{Z} : (x+5)(3x-1) = 0\}$ and solving the quadratic in this set yields $x = -5,\ x = \frac{1}{3}$. Only -5 is an integer so

$$C = \{-5\}\,.$$

(d) This time we have the same quadratic in part (c) so we have the same solution $x = -5,\ x = \frac{1}{3}$ but universal set is the rationals $\mathbb{Q}$ so

$$D = \left\{\frac{1}{3},\ -5\right\}$$

(e) The given linear equation $x - \pi = 0 \ \Rightarrow\ x = \pi$. We haven't shown this but π is *not* a rational number so $E = \varnothing$.

(f) We have the same equation as part (e) but the universal set is $\mathbb{R}$ and π is a real number so $F = \{\pi\}$.

3. Using the set notation, we have the following:

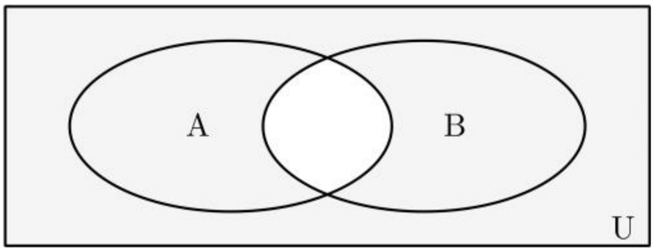
(a) $\{x \in \mathbb{R} : x < 0\}$ (b) $\{x \in \mathbb{Z} : x > 0\}$ or $\mathbb{N}$ (c) $\{x \in \mathbb{R} : 0 < x < 2\}$

(d) $\{x \in \mathbb{Q} : x < 1\}$ (e) $\{10n : n \in \mathbb{N}\}$

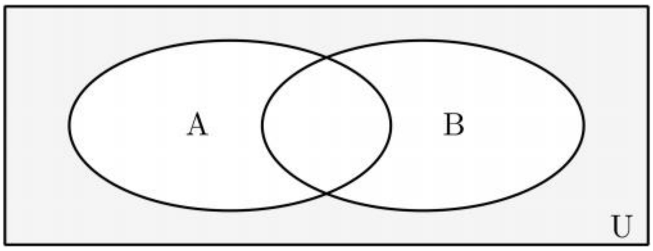
4. (a) See Fig. 12.

(b) Same as Fig. 12 with the set labelled B instead of A.

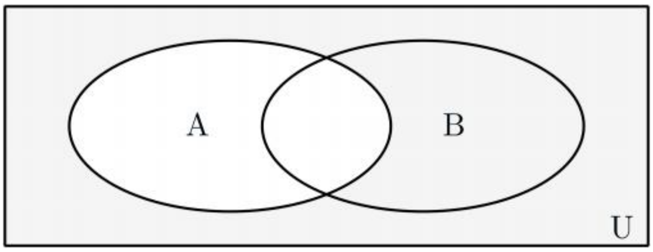
(c) The Venn diagram for $(A \cap B)^c$ is shaded below:



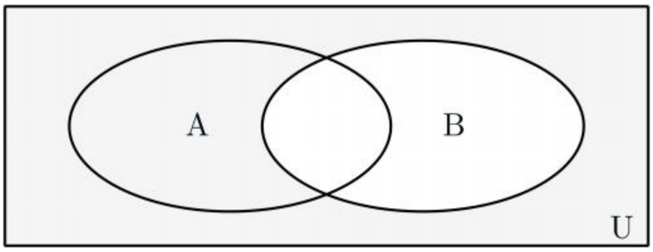
(d) The Venn diagram for $(A \cup B)^c$ is shaded:



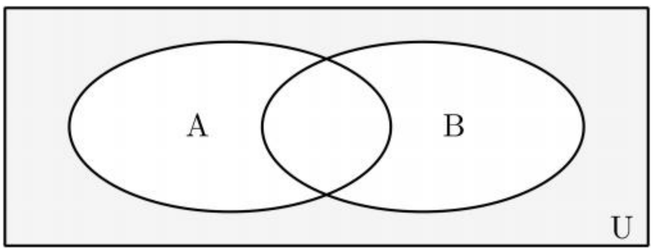
(e) The Venn diagram for $A^c \cap B^c$ means we first shade outside of the set A:



Now we shade outside of B:



The shading overlapping of the last two Venn diagrams gives us everything outside of $A \cup B$ which is:

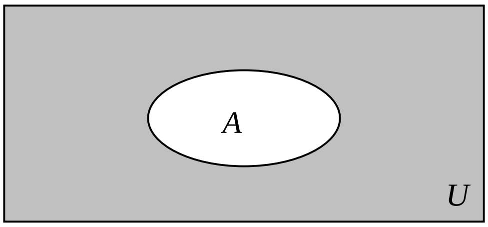


Notice that according to these Venn diagrams we have

$$\left(A \cup B\right)^c = A^c \cap B^c .$$

It can be shown that this result always holds, that is $\left(A \cup B\right)^c = A^c \cap B^c$.

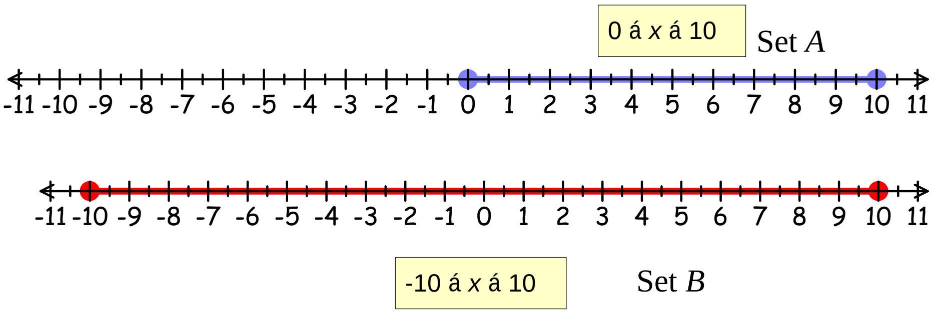
5. From Fig. 12 we have



A^c

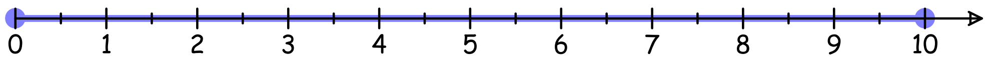
Now for $\left(A^c\right)^c$ we only shade the blank part in this Venn diagram which gives us the set A. Hence $\left(A^c\right)^c = A$.

6. (a) We can draw the given sets $A = \left\{x \in \mathbb{R} : 0 \le x \le 10\right\}$ and $B = \left\{x \in \mathbb{R} : -10 \le x \le 10\right\}$ on the number line as follows:



Clearly the set A is a subset of B, that is $A \subseteq B$.

(b) Again, drawing the given sets $A = \left\{x \in \mathbb{Z} : 0 \le x \le 10\right\}$ and $B = \left\{x \in \mathbb{R} : 0 \le x \le 10\right\}$ we have



The set A is the set of all the integers (whole numbers) between 0 to 10, that is $A = \left\{0,\ 1,\ 2,\ \cdots,\ 10\right\}$ whilst the set B is all the real numbers between 0 to 10, that is all the line in the above diagram. Again, the set A is a subset of the set B, that is $A \subseteq B$.

(c) This is similar to the sets in part (b) but we have switched sets A and B. Therefore, we have $B \subseteq A$.

(d) *What does the set notation* $A = \{x \in \mathbb{N} : -10 \leq x \leq 10\}$ *mean?*

It means that x is a natural number between -10 to 10. *What numbers does this set include?*

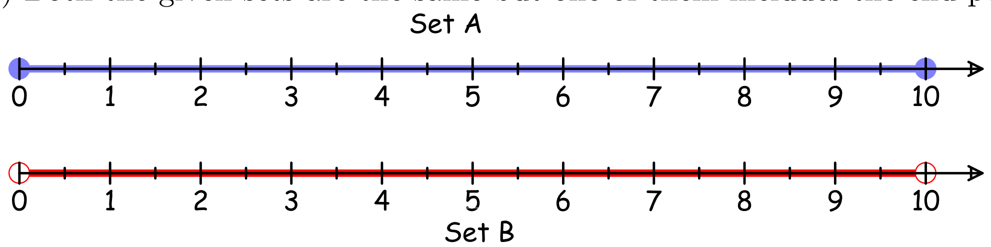
Remember natural numbers are whole numbers greater than or equal to 1, therefore $A = \{1,\ 2,\ 3,\ \cdots,\ 10\}$. *What does* $B = \{x \in \mathbb{Z} : 1 \leq x \leq 10\}$ *represent?*

It is the same as the set A because we have integers between 1 to 10, that is

$$B = \{1,\ 2,\ 3,\ \cdots,\ 10\}.$$

Hence we have $A \subseteq B$ and $B \subseteq A$ which means we have $A = B$.

(e) Both the given sets are the same but one of them includes the end points:



Clearly $B \subseteq A$ because the set B does *not* include the end points 0 and 10.

7. Before we evaluate which of the given sets are subsets, we need to write down the members of each given set:

$$\varnothing,\ A = \{2,\ 3,\ 5\},\ B = \{x \in \mathbb{Z} : x^2 - 4 = 0\}$$
$$C = \{x \in \mathbb{N} : x \text{ is prime and less than } 10\},\ D = \{x \in \mathbb{Z} : 0 \leq x \leq 10\}$$

Then we check for subsets.

(a) For the given sets $\varnothing$ and A we have $\varnothing \subseteq A$ because the empty set $\varnothing$ is a subset of every set.

(b) *Is A subset of A?*

Yes we have $A \subseteq A$. [Every set is a subset of itself].

(c) We need to examine the given sets $A = \{2,\ 3,\ 5\}$ and $C = \{2,\ 3,\ 5,\ 7\}$.

Clearly all the elements of the set A which are 2, 3 and 5 are in the set C therefore A is a subset of C, that is $A \subseteq C$.

(d) Similarly, we have $C = \{2, 3, 5, 7\}$ and $D = \{0, 1, 2, 3, \cdots, 10\}$.

Again all the elements of the set C which are 2, 3, 5 and 7 are in the set D therefore C is a subset of D, that is $C \subseteq D$.

(e) *Is the set B subset of the set C?*

We have $B = \{-2, 2\}$ and $C = \{2, 3, 5, 7\}$ but the member -2 in the set B is *not* in the set C therefore $B \not\subseteq C$. *What does this notation mean?*

The set B is not a subset of the set C.

8. We need to write out the elements of each of the given sets.

$$A = \{x \in \mathbb{Z} : 0 < x < 5\}, \quad B = \{x \in \mathbb{N} : x \text{ is an even number}\}$$
$$C = \{x \in \mathbb{N} : x \text{ is a multiple of } 2\}$$
$$D = \{x \in \mathbb{R} : x \neq x\}, \quad E = \{x \in \mathbb{N} : x^3\}$$
$$F = \{x \in \mathbb{Z} : 0 < x < 2\}$$

Note that the set D is the empty set because there is no real number x such that $x \neq x$.

(a) We have $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 6, 8, \cdots\}$ therefore $A \not\subseteq B$ because the elements 1, 3, 5, ... are *not* in the set B.

(b) Similarly, we have $A = \{1, 2, 3, 4\}$ and $C = \{2, 4, 6, 8, \cdots\}$ therefore $A \not\subseteq C$.

(c) We have $B = \{2, 4, 6, 8, \cdots\}$ and $C = \{2, 4, 6, 8, \cdots\}$ which means we have $B \subseteq C$.

(d) B and C are the same sets as in part (c), that is

$$B = \{2, 4, 6, 8, \cdots\} \text{ and } C = \{2, 4, 6, 8, \cdots\}$$

therefore $C \subseteq B$. In fact, $B = C$.

(e) We have $D = \varnothing$ and $A = \{1, 2, 3, 4\}$ therefore the set A cannot be a subset of the empty set $\varnothing$ which means we have $A \not\subseteq D$.

(f) Because $D = \varnothing$ and the empty set is a subset of every set therefore $D \subseteq A$.

(g) We have the sets $E = \{1, 8, 27, 64, \cdots\}$ and $F = \{1\}$. Thus the set $E = \{1, 8, 27, 64, \cdots\}$ *cannot* be a subset of the set $F = \{1\}$. We have $E \not\subseteq F$.

(h) As part (g) we have $E = \{1,\ 8,\ 27,\ 64,\ \cdots\}$ and $F = \{1\}$ and since the member 1 is in the set $E = \{1,\ 8,\ 27,\ 64,\ \cdots\}$ therefore $F \subseteq E$.

9. *What does the term cardinality mean?*

Cardinality is the number of elements in the set and is denoted by $|A|$.

(a) Since $\varnothing$ denotes the empty set which means it has *no* elements therefore

$$|\varnothing| = 0.$$

(b) We are given the set $A = \{a,\ b,\ c\}$ therefore $|A| = 3$ because the set has 3 members.

(c) We are given $A = \{x \in \mathbb{Z} : 3x^2 - x = 0\}$ and we need to find the elements of this set A. Solving the given quadratic

$$\begin{aligned} 3x^2 - x &= 0 \\ x(3x - 1) &= 0 \qquad \text{[Factorising]} \\ x = 0,\ x &= \frac{1}{3} \end{aligned}$$

Since $x \in \mathbb{Z}$ therefore members of this set A can only be integers (whole numbers) which means only 0 is a member. Thus $A = \{0\}$. *What is the cardinality of this set?*

Since the set is a singleton (only one element) therefore the cardinality $|A| = 1$.

(d) *What are the elements of the given set* $A = \{x \in \mathbb{R} : x = x + 1\}$ *?*

For any real number x we have $x \neq x + 1$ therefore there are no x values which satisfy the equation $x = x + 1$. Hence the set A is empty, that is $A = \varnothing$ and so $|A| = 0$.

10. We are given the sets

$$A = \{1,\ 2,\ 3,\ 4,\ 5\} \text{ and } B = \{x \in \mathbb{N} : x \text{ is a prime number } \leq 5\}.$$

What are the elements of the set B?

$$B = \{x \in \mathbb{N} : x \text{ is a prime number } \leq 5\} = \{2, 3, 5\}.$$

How can we show $A \not\subseteq B$?

Recall by Definition (I.22) we have $A \subseteq B$ if every element of set A is also in the set B. In this case we have $1 \in A$ but $1 \notin B$ therefore $A \not\subseteq B$.

How do we show $B \subseteq A$?

Again, by using the definition (I.22) we show that every element of the set B is also in the set A. We have $B = \{2,\ 3,\ 5\}$ and all three elements 2, 3 and 5 are in the set $A = \{1,\ 2,\ 3,\ 4,\ 5\}$ therefore $B \subseteq A$.

11. We have $A = \{1,\ 3\}$, $B = \{1,\ 3,\ 3,\ 1\}$ and $C = \left\{1,\ 3,\ \frac{3}{1},\ \frac{\pi}{\pi}\right\}$.

Remember from main text that a set such as $\{a,\ b\}$ is the same as $\{a,\ b,\ a,\ b\}$. Here we have

$$B = \{1,\ 3,\ 3,\ 1\} = \{1,\ 3\}$$

$$C = \left\{1,\ 3,\ \frac{3}{1},\ \frac{\pi}{\pi}\right\} = \{1,\ 3,\ 3,\ 1\} = \{1,\ 3\}$$

Thus, we have $A = B = C = \{1,\ 3\}$.

12. *Proof.* (By Contradiction).

Suppose there is an integer n such that

$$\sum_{m=1}^{n} m = 1 + 2 + 3 + 4 + \ldots + n \neq \frac{n(n+1)}{2} \qquad (*)$$

Consider the set S given by

$$S = \left\{n \in \mathbb{N} : 1 + 2 + \cdots + n \neq \frac{n(n+1)}{2}\right\}$$

By (*) the set S is non – empty. By the Well Ordering Principle (WOP) there is a least element, say m, which is a member of the set S. Note that $m > 1$ because for $n = 1$ we have our given result. Clearly $m - 1 \notin S$ because m is the least positive integer in S. This implies the given proposition is true of $m - 1$:

$$1 + 2 + 3 + \ldots + (m-1) = \frac{(m-1)(m-1+1)}{2} = \frac{m(m-1)}{2} \qquad (**)$$

Using this (**) to find the sum of the first m terms gives

$$1 + 2 + 3 + \ldots + (m-1) + m = \frac{m(m-1)}{2} + m$$

$$= \frac{m^2 - m + 2m}{2} = \frac{m^2 + m}{2} = \frac{m(m+1)}{2}$$

This implies the given proposition is true when $n = m$. Therefore, m *cannot* be the least positive integer where the given proposition is false. This is a contradiction so,

there is *no* least integer where the given proposition is false.

■

13. We need to prove

Principle of Mathematical Induction (I.15)
For each natural number n, let $P(n)$ be a proposition about n. If $P(n)$ satisfies: 1) $P(1)$ is true, 2) For an arbitrary k, $P(k)$ is true implies $P(k+1)$ is true. Then for *all* natural numbers, n, we have $P(n)$ is true.

By using the WOP.

Proof.

Suppose the result is *not* true for *all* the natural numbers. There is n such that $P(n)$ is false. Let S be the subset of natural numbers where $P(n)$ is false. Then S is non – empty. By the Well Ordering Principle:

(I.24) Every non-empty subset of positive integers has a *least* element.

The set S has a lease element. Let ℓ be this least element. Then $1 \notin S$ therefore $\ell > 1$ which implies that $\ell - 1 > 0$ is a natural number. However $\ell - 1$ *cannot* be in S because $\ell - 1 < \ell$ and ℓ is the least element of S. With $\ell - 1 \notin S$ which implies that $P(n)$ is true for $n = \ell - 1$. Applying step 2 of the induction principle (I.15) on $n = \ell - 1$ gives that

$$n + 1 = \ell - 1 + 1 = \ell$$

Therefore the $P(\ell)$ is true but this is impossible because ℓ is in S which implies that $P(\ell)$ is false. We have a contradiction, that is S is an empty set and the result is true for all natural numbers. This completes our proof.

■

Complete Solutions to Exercises I.6

1. *Proof.* Since $a < b$ which means that $a - b < 0$ and so

$$\begin{aligned} a - b &= a \underbrace{-c + c}_{=0} - b \\ &= a - c - \left(b - c\right) < 0 \end{aligned}$$

From $a - c - \left(b - c\right) < 0$ we have the required result, $a - c < b - c$.

■

2. *Proof.* From the two given inequalities, $a < b$ and $b < c$, we have

$$a - b < 0 \text{ and } b - c < 0 \text{ respectively.}$$

Adding these inequalities, $a - b < 0$ and $b - c < 0$, we have

$$a \underbrace{-b + b}_{=0} - c < 0$$

$$a - c < 0 \quad \text{which implies} \quad a < c$$

Hence, we have $a < c$ which is what we were trying to prove.

■

3. *Proof.* From the two given inequalities, $a < b$ and $c \le d$, we have

$$a + c \le a + d < b + d$$

Hence, we have our result, $a + c < b + d$.

■

4. (a) *Proof.* We use prove by contradiction. Suppose $\frac{1}{x} < 0$. Multiplying through by $x^2 > 0$ we have

$$\frac{1}{x}\left(x^2\right) < 0\left(x^2\right)$$

$$x < 0$$

$x < 0$ (x is less than 0) is a contradiction because we are given $x > 0$ (x is greater than 0). Hence our supposition $\frac{1}{x} < 0$ is wrong therefore we have our result, $\frac{1}{x} > 0$.

■

(b) *Proof.* Suppose $\frac{1}{x} > 0$. Multiplying through by $x^2 > 0$ we have

$$\frac{1}{x}\left(x^2\right) > 0\left(x^2\right)$$

$$x > 0$$

$x > 0$ (x is greater than 0) is a contradiction because we are given $x < 0$ (x is less than 0). Hence our supposition $\frac{1}{x} > 0$ is wrong therefore we have our result, $\frac{1}{x} < 0$.

■

5. (a) *Proof.* We use proof by contradiction. Suppose $\frac{1}{a} < \frac{1}{b}$. From this we have

$$\begin{aligned} \frac{1}{a} - \frac{1}{b} &< 0 \\ b - a &< 0 \qquad \left[\text{Multiplying through by } ab\right] \\ b &< a \end{aligned}$$

Remember $b < a \Leftrightarrow a > b$. Contradiction! *How?*

Because we are given $a < b$ (a is less than b) and we have deduced $a > b$ (a is greater than b). Our supposition $\frac{1}{a} < \frac{1}{b}$ must be false. Therefore $\frac{1}{a} \geq \frac{1}{b}$. But $\frac{1}{a} \neq \frac{1}{b}$ because $a \neq b$. Hence, we have the strict inequality, $\frac{1}{a} > \frac{1}{b}$.

■

(b) *Proof.* We use proof by contradiction. Suppose $\frac{1}{b} < \frac{1}{a}$. From this we have

$$\begin{aligned} \frac{1}{b} - \frac{1}{a} &< 0 \\ a - b &< 0 \qquad \left[\text{Multiplying through by } ab\right] \\ a &< b \end{aligned}$$

Recall $a < b \Leftrightarrow b > a$. Contradiction! *How?*

Because we are given $b < a$ (b is less than a) and we have deduced $b > a$ (b is greater than a). Our supposition $\frac{1}{b} < \frac{1}{a}$ must be false. Therefore $\frac{1}{b} \geq \frac{1}{a}$. But $\frac{1}{a} \neq \frac{1}{b}$ because $a \neq b$. Hence, we have $\frac{1}{b} > \frac{1}{a}$.

■

6. (a) *Proof.* We have $0 < a < b$ and if $x = 0$ then $x^2 = 0$, therefore $ax^2 = bx^2 = 0$. If $x \neq 0$ then by Proposition (I.30) we have $x^2 > 0$ and so $ax^2 < bx^2$. Hence, we have the result that we are trying to prove, $ax^2 \leq bx^2$.

■

(b) *Proof.* If $x = 0$ then $x^2 = 0$, therefore $ax^2 = bx^2 = 0$. If $x \neq 0$ then by Proposition (I.30) $x^2 > 0$ and multiplying this by -1 gives

$$\begin{aligned} (-1)x^2 &< (-1)0 && [\text{Change Inequality because } -1 < 0] \\ -x^2 &< 0 \end{aligned}$$

Multiplying the given inequality, $a < b$, by $-x^2$ we have

$$\begin{aligned} a(-x^2) &> b(-x^2) && [\text{Change Inequality because } -x^2 < 0] \\ -ax^2 &> -bx^2 && [\text{Because } a(-x^2) = -ax^2 \text{ and } b(-x^2) = -bx^2] \end{aligned}$$

which is the result we are trying to prove, $-ax^2 \geq -bx^2$.

■

7. *Proof.* We have

$$x^2 - 2x + 1 = (x-1)^2 \geq 0 \qquad [\text{By the Proposition (I.30)}]$$

■

8. *Proof.* We have

$$\begin{aligned} x^2 - 5x + 9 &= \left(x - \frac{5}{2}\right)^2 - \frac{25}{4} + 9 && [\text{Completing the Square}] \\ &= \left(x - \frac{5}{2}\right)^2 - \frac{25}{4} + \frac{36}{4} && \left[\text{Because } 9 = \frac{36}{4}\right] \\ &= \left(x - \frac{5}{2}\right)^2 + \frac{11}{4} \\ &\geq 0 + \frac{11}{4} = \frac{11}{4} && \left[\text{Because } \left(x - \frac{5}{2}\right)^2 \geq 0\right] \end{aligned}$$

■

9. *Proof.* We have

$$\begin{aligned} (a+b)^2 &= a^2 + 2ab + b^2 \\ &= a^2 + b^2 + 2ab && (*) \end{aligned}$$

We need to prove $a^2 + b^2 \geq 2ab$. *How?*

Consider $(a-b)^2$. By Proposition (I.30) we have $(a-b)^2 \geq 0$ and

$$(a-b)^2 = a^2 + b^2 - 2ab \geq 0 \text{ which implies that } a^2 + b^2 \geq 2ab$$

Substituting this inequality, $a^2 + b^2 \geq 2ab$, into (*) gives

$$\begin{aligned} (a+b)^2 &= a^2 + b^2 + 2ab \\ &\geq 2ab + 2ab = 4ab \end{aligned}$$

This, $(a+b)^2 \geq 4ab$, is the required result.

■

10. *Proof.* We have

$$\left[\frac{1}{2}\left(x+y\right)\right]^2 = \frac{1}{4}\left(x^2+y^2\right)+\frac{1}{2}xy \qquad (\dagger)$$

We are required to prove that $\frac{1}{2}xy \le \frac{1}{4}\left(x^2+y^2\right)$. *How?*

Consider $\left[\frac{1}{2}\left(x-y\right)\right]^2$. By Proposition (I.30) we have $\left[\frac{1}{2}\left(x-y\right)\right]^2 \ge 0$ and

$$\begin{aligned}\left[\frac{1}{2}\left(x-y\right)\right]^2 &= \frac{1}{4}\left(x^2-2xy+y^2\right) \qquad \left[\text{Expanding}\right]\\ &= \frac{1}{4}\left(x^2+y^2\right)-\frac{1}{2}xy \ge 0\\ \frac{1}{4}\left(x^2+y^2\right) &\ge \frac{1}{2}xy\end{aligned}$$

We can write the last line, $\frac{1}{4}\left(x^2+y^2\right) \ge \frac{1}{2}xy$, as $\frac{1}{2}xy \le \frac{1}{4}\left(x^2+y^2\right)$ and substituting this into (†) gives

$$\begin{aligned}\left[\frac{1}{2}\left(x+y\right)\right]^2 &= \frac{1}{4}\left(x^2+y^2\right)+\frac{1}{2}xy\\ &\le \frac{1}{4}\left(x^2+y^2\right)+\frac{1}{4}\left(x^2+y^2\right)\\ &= \frac{1}{2}\left(x^2+y^2\right)\end{aligned}$$

We have proven our result, $\left[\frac{1}{2}\left(x+y\right)\right]^2 \le \frac{1}{2}\left(x^2+y^2\right)$.

■

11. We use Definition (I.32) which is $\left|x\right| = \begin{cases} x & \text{if } x \ge 0 \\ -x & \text{if } x < 0 \end{cases}$ in each case:

$$\left|e\right| = e,\ \left|-e\right| = -\left(-e\right) = e\,,$$

$$\left|-\sqrt{2}\right| = -\left(-\sqrt{2}\right) = \sqrt{2}\,,$$

$$\left|-6-7\right| = \left|-13\right| = -\left(-13\right) = 13\,,$$

$$\left|\cos\left(\frac{3\pi}{4}\right)\right| = \left|-\frac{1}{\sqrt{2}}\right| = -\left(-\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}\,.$$

12. We need to solve inequalities using the modulus function. In each case we use (I.34) $\left|x\right| < a \Leftrightarrow -a < x < a$.

a. We are given $\left|x\right| < 1$ therefore by (I.34) we have

$$\left|x\right| < 1 \Leftrightarrow -1 < x < 1.$$

b. Similarly, we have $\left|x\right| < \pi \Leftrightarrow -\pi < x < \pi$.

c. Now we are given $\left|x-1\right| < 1$ so using (I.34) we have

$$\begin{aligned}\left|x-1\right| < 1 &\Leftrightarrow -1 < x-1 < 1\\ &\Leftrightarrow -1+1 < x < 1+1\\ &\Leftrightarrow 0 < x < 2\end{aligned}$$

d. Similarly we have

$$\begin{aligned}\left|x-5\right| \le 2 &\Leftrightarrow -2 \le x-5 \le 2\\ &\Leftrightarrow -2+5 < x < 2+5\\ &\Leftrightarrow 3 < x < 7\end{aligned}$$

13. We need to prove $\left|x-y\right| = \left|y-x\right|$.

Proof.

We use Proposition (I.35) $\left|xy\right| = \left|x\right|\left|y\right|$ on this. We have

$$\begin{aligned}\left|x-y\right| &= \left|-(y-x)\right|\\ &= \left|-1(y-x)\right|\\ &= \left|-1\right|\left|y-x\right| \qquad \left[\text{By (I.35)}\right]\\ &= 1\left|y-x\right| = \left|y-x\right|\end{aligned}$$

Therefore $\left|x-y\right| = \left|y-x\right|$.

■

14. We are asked to prove $\left|\dfrac{1}{x}\right| = \dfrac{1}{\left|x\right|}$ where $x \ne 0$.

Proof.

We use Proposition (I.35) $\left|xy\right| = \left|x\right|\left|y\right|$:

$$\left|\frac{1}{x}\right| = \left|1\times\frac{1}{x}\right| = \left|1\right|\times\left|\frac{1}{x}\right| = 1\times\frac{\left|1\right|}{\left|x\right|} = \frac{1}{\left|x\right|}.$$

This completes our proof.

■

15. We are asked to prove $\left|\dfrac{x}{y}\right| = \dfrac{\left|x\right|}{\left|y\right|}$ where $y \ne 0$.

Proof.

We use Proposition (I.35) $|xy| = |x||y|$ and the result of the previous question we have

$$\left|\frac{x}{y}\right| = \left|x \times \frac{1}{y}\right|$$
$$= |x| \times \left|\frac{1}{y}\right| = |x| \times \frac{1}{|y|} = \frac{|x| \times 1}{|y|} = \frac{|x|}{|y|}$$

This finishes our proof.

■

16. We are required to prove $\left|\frac{1}{n}\right| = \frac{1}{n}$.

Proof.

We use Proposition (I.35) $|xy| = |x||y|$ and also $\left|\frac{1}{n}\right| = \frac{1}{n}$ because $n \in \mathbb{N}$ and the natural numbers are positive. We have

$$\left|\frac{1}{n}\right| = \left|1 \times \frac{1}{n}\right| = |1| \times \left|\frac{1}{n}\right| = 1 \times \frac{1}{n} = \frac{1}{n}.$$

This completes our proof.

■

17. We can disprove $(n+1)^2 \geq 2n^2$ by giving a counter example. Consider $n = 3$, therefore

$$(3+1)^2 = 16 \leq 2(3^2) = 18.$$

(Challenge). Proof. We use proof by induction because the given result concerns natural numbers. The procedure for induction is to prove the result for $n = 3$, assume it is true for $n = k$ and then prove it for $n = k+1$. For $n = 3$ we have

$$(3+1)^2 = 16 \leq 2(3^2) = 18. \qquad \surd$$

Assume the result is true for $n = k$ that is

$$(k+1)^2 \leq 2k^2 \qquad (*)$$

Required to prove

$$(k+1+1)^2 \leq 2(k+1)^2$$

How?

By expanding the left - hand side and using the result $(k+1)^2 \le 2k^2$:

$$\begin{aligned}(k+1+1)^2 &= (k+2)^2 \\ &\le k^2+4k+4 \\ &= \underbrace{k^2+2k+1}_{=(k+1)^2}+2k+3 \\ &= (k+1)^2+2k+3\end{aligned}$$

Using (*) we have

$$\begin{aligned}(k+1+1)^2 &= (k+1)^2+2k+3 && \text{[From Above]} \\ &\le 2k^2+2k+3 && \text{[By (*)]} \\ &\le 2k^2+4k+2 = 2(k+1)^2\end{aligned}$$

Hence, we have our result for $n=k+1$. Therefore, by induction we have proven $(n+1)^2 \le 2n^2$. *(End of Challenge).*

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18. We need to evaluate $\prod_{j=1}^{5}\left(\frac{2j-1}{2j}\right)\left(\frac{2j+1}{2j}\right)$. Note that the numerator is difference of two squares and the denominator is square term. We have

$$\left(\frac{2j-1}{2j}\right)\left(\frac{2j+1}{2j}\right) = \frac{4j^2-1}{4j^2}$$

By substituting $j = 1, 2, 3, 4$ and 5 we have

$$\begin{aligned}\prod_{j=1}^{5}\left(\frac{2j-1}{2j}\right)\left(\frac{2j+1}{2j}\right) &= \prod_{j=1}^{5}\left(\frac{4j^2-1}{4j^2}\right) \\ &= \left(\frac{4-1}{4}\right)\times\left(\frac{4(2^2)-1}{4(2^2)}\right)\times\left(\frac{4(3^2)-1}{4(3^2)}\right)\times\left(\frac{4(4^2)-1}{4(4^2)}\right)\times\left(\frac{4(5^2)-1}{4(5^2)}\right) \\ &= \frac{3}{4}\times\frac{15}{16}\times\frac{35}{36}\times\frac{63}{64}\times\frac{99}{100} = \frac{9\ 823\ 275}{14\ 745\ 600} = 0.666\ (3\text{sf})\end{aligned}$$

We also need to find $\left|\frac{2}{\pi}-\prod_{j=1}^{5}\left(\frac{2j-1}{2j}\right)\left(\frac{2j+1}{2j}\right)\right|$. By the above result we have

$$\left|\frac{2}{\pi}-\prod_{j=1}^{5}\left(\frac{2j-1}{2j}\right)\left(\frac{2j+1}{2j}\right)\right| = \left|\frac{2}{\pi}-0.666\right| = |-0.0296| = 0.0296\ (3\text{sf})$$

19. Here we only give some of the complete solutions as the method is identical with different numbers.

(a) We are given $x^2 - 4x + 3$ and so we have

$$x^2 - 4x + 3 = (x-2)^2 - 4 + 3 = (x-2)^2 - 1$$

(b) Similarly, for $x^2 + 7x + 1$ we have

$$x^2 + 7x + 1 = \left(x + \frac{7}{2}\right)^2 - \left(\frac{7}{2}\right)^2 + 1$$

$$= \left(x + \frac{7}{2}\right)^2 - \frac{49}{4} + \frac{4}{4} = \left(x + \frac{7}{2}\right)^2 - \frac{45}{4}$$

(e) We write the given quadratic polynomial $9 + 8x - x^2$ as

$$9 + 8x - x^2 = 9 - \left(x^2 - 8x\right)$$

$$= 9 - \left[(x-4)^2 - 16\right] = 9 - (x-4)^2 + 16 = 25 - (x-4)^2$$

(f) We take out a factor of 3 from the given quadratic polynomial

$3x^2 + 7x + 1$:

$$3x^2 + 7x + 1 = 3\left(x^2 + \frac{7}{3}x + \frac{1}{3}\right)$$

$$= 3\left[\left(x + \frac{7}{6}\right)^2 - \left(\frac{7}{6}\right)^2 + \frac{1}{3}\right]$$

$$= 3\left[\left(x + \frac{7}{6}\right)^2 - \frac{49}{36} + \frac{12}{36}\right] = 3\left[\left(x + \frac{7}{6}\right)^2 - \frac{37}{36}\right]$$

Now taking the 3 in the last term on the right we have

$$3x^2 + 7x + 1 = 3\left(x + \frac{7}{6}\right)^2 - 3\left(\frac{37}{36}\right) = 3\left(x + \frac{7}{6}\right)^2 - \frac{37}{12}$$

(ii) All the hard work has been done in part (a). If we repeat the technique given in Example 44 we have:

(a) $y = x^2 - 4x + 3 = (x-2)^2 - 1 \ge 0 - 1 = -1$ gives

$$\min\left\{y \in \mathbb{R} : y = x^2 - 4x + 3\right\} = -1$$

(b) $y = x^2 + 7x + 1 = \left(x + \frac{7}{2}\right)^2 - \frac{45}{4} \ge 0 - \frac{45}{4} = -\frac{45}{4}$ we obtain

$$\min\left\{y \in \mathbb{R} : y = x^2 + 7x + 1\right\} = -\frac{45}{4}$$

(e) $y = 9 + 8x - x^2 = 25 - (x-4)^2 \le 25 - 0 = 25$. So, we have

$$\max\left\{y \in \mathbb{R} : y = 9 + 8x - x^2\right\} = 25$$

(f) $y = 3x^2 + 7x + 1 = 3\left(x + \frac{7}{6}\right)^2 - \frac{37}{12} \geq 0 - \frac{37}{12} = -\frac{37}{12}$. Hence

$$\min\left\{y \in \mathbb{R} : y = 3x^2 + 7x + 1\right\} = -\frac{37}{12}$$

20. The complete solutions are at the following url:

Complete solutions to question 20

You need to look at solutions to question 2 and for part (l) the solution is to question 4.