Exercise 2e

Throughout this exercise $\sum$ represents $\sum_{n=1}^{\infty}$.

1. Discuss convergence or divergence for each of the following series:

   (a) $\sum\left(\frac{1}{(2n)!}\right)$ (b) $\sum_{n=0}^{\infty}\left(\frac{n!}{2^n}\right)$ (c) $\sum_{n=0}^{\infty}\left(\frac{n!}{3^n}\right)$

   (d) $\sum\left(\frac{(n+1)^2}{2^n}\right)$ (e) $\sum\left(e^{-n}\right)$ (f) $\sum\left(\frac{(n+1)^2}{3^n}\right)$

   (g) $\sum\left(\frac{10^n}{n!}\right)$ (h) $\sum\left(\frac{(n!)^2}{(2n)!}\right)$ (i) $\sum\left(\frac{\left(\sqrt{7}-1\right)^n}{n^2+1}\right)$

   (j) $\sum\left(\frac{3^n n}{(n+1)^2}\right)$ (k) $\sum\left(\frac{n!}{(2n+1)!}\right)$ (m) $\sum\left(\frac{5^n}{2^{n+1}n}\right)$

   (n) $\sum\left(n^{-n}\right)$ (o) $\sum\left(n^n e^{-n}\right)$

2. (a) Discuss convergence or divergence for each of the following series:

   (i) $\sum\left(\frac{2^n n!}{n^n}\right)$ (ii) $\sum\left(\frac{3^n n!}{n^n}\right)$

   (b) Determine the values of x for which the following series

   $$\sum\left(\frac{x^n n!}{n^n}\right)$$

   (i) converges (ii) diverges

   $\left(\text{Hint: } \lim_{n\to\infty}\left(\frac{n}{1+n}\right)^n = e\right)$

3. Test the following series for convergence $\sum\left(\frac{n^n}{n!}\right)$.

4. (i) Show that the ratio test fails for each of the following series:

   (a) $\sum\left(\frac{1}{n^3}\right)$ (b) $\sum\left(\frac{1}{n+1}\right)$ (c) $\sum\left(\frac{1}{n^2+1}\right)$

   (d) $\sum\left(\frac{1}{\sqrt{n^2+1}}\right)$ (e) $\sum\left(\frac{1}{\sqrt{n+1}}\right)$ (f) $\sum\left(\frac{1}{n^p}\right)$

   (ii) Test each of the series in part (a).

Solutions

1. (a) Converges because $L = 0$.
   (a) Diverges because $L = +\infty$.
   (b) Diverges because $L = +\infty$.
   (c) Converges because $L = 1/2$.
   (d) Converges because $L = 1/e$.
   (e) Converges because $L = 1/3$.
   (f) Converges because $L = 0$.
   (g) Converges because $L = 1/4$.
   (h) Diverges because $L = \sqrt{7} - 1$.
   (i) Diverges because $L = 3$.
   (j) Converges because $L = 0$.
   (k) Converges because $L = 2/3$.
   (l) Diverges because $L = 5/2$.
   (m) Converges because $L = 0$.
   (n) Diverges because $L = +\infty$.

2. (a) (i) $L = \frac{2}{e} < 1$ series converges (ii) Diverges because $L = \frac{3}{e} > 1$.
(b) (i) $0 < x < e$ (ii) $x > e$

3. Diverges because $L = e > 1$.

4. (i) In each case $L = 1$.
(ii) (a) Converges because $p = 3 > 1$.
(b) Diverges. Use the limit comparison test and compare with $1/n$.
(c) Converges. Use the normal comparison test with $1/n^2$.
(d) Diverges. Use the limit comparison test and compare with $1/n$.
(e) Diverges. Use the limit comparison test and compare with $1/\sqrt{n}$.
(f) This is the p-series test with convergence for $p > 1$ and divergence for $p \le 1$.