Section D: Taylor Series of Functions

By the end of this section you will be able to

- understand why we need Taylor series
- determine the Taylor series of functions
- find the remainder term

D1 Determining the Taylor Series

From our previous studies we know we can obtain a Maclaurin series for basic functions such as $\sin(x)$, $\cos(x)$ and e^x. *Why can't we find a Maclaurin series for the logarithmic function* $\ln(x)$*?*

This is because $\ln(x)$ is **not** defined at $x = 0$. Recall that the graph of $\ln(x)$ is:

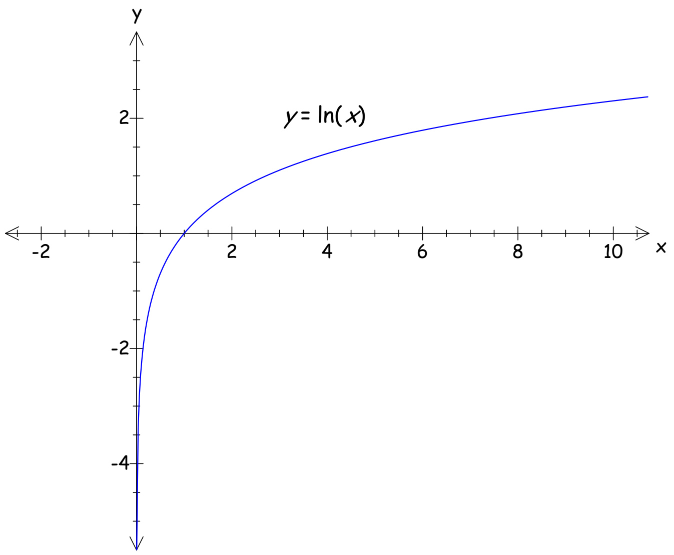

Fig 19

$\ln(x)$ is **not** defined for any negative or zero *x* values. This means that we **cannot** find a Maclaurin series for $\ln(x)$ because the Maclaurin series is defined for functions that can be differentiated at $x = 0$. *Can we find a series representation for* $\ln(x)$ *at another point, say* $x = a$*?*

Yes and such a series is called a **Taylor series**.

The Taylor series has the same pattern as the Maclaurin series because it is derived in the same manner.

The Taylor series for a function $f(x)$ about a point $x=a$ is given by:

(3.8) $$f(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\ldots$$

Provided the given function $f(x)$ is $(n+1)$ times differentiable on an open interval containing the points *a* and *x*.

We can only find the Taylor series if $f(x)$ and its derivatives are defined at $x=a$ and the infinite series approaches a finite value. *What happens at* $a=0$ *?*

Substituting $a=0$ into the above Taylor series (3.8) gives the Maclaurin series.

Example 17

Obtain the Taylor series for $\ln(x)$ about the point $a=2$.

Solution

Let $f(x)=\ln(x)$. *How do we find the Taylor series for* $f(x)=\ln(x)$ *about* $a=2$ *?*

Need to differentiate $\ln(x)$ and then substitute $x=2$ into our result:

$$f(x)=\ln(x) \qquad\qquad f(2)=\ln(2)$$

$$f'(x)=\frac{1}{x}=x^{-1} \quad [\text{Differentiating}] \quad f'(2)=\frac{1}{2}$$

$$f''(x)=-x^{-2}=-\frac{1}{x^2} \quad [\text{Differentiating}] \quad f''(2)=-\frac{1}{2^2}=-\frac{1}{4}$$

$$f'''(x)=2x^{-3}=\frac{2}{x^3} \quad [\text{Differentiating}] \quad f'''(2)=\frac{2}{2^3}=\frac{1}{4}$$

$$f^{(4)}(x)=-6x^{-4}=-\frac{6}{x^4} \quad [\text{Differentiating}] \quad f^{(4)}(2)=-\frac{6}{2^4}=-\frac{3}{8}$$

Substituting these values into the general Taylor series

(3.8) $$f(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\ldots$$

with $a=2$ and $f(x)=\ln(x)$ we have

$$\ln(x) = f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3 + \frac{f^{(4)}(2)}{4!}(x-2)^4 + \dots$$
$$= \ln(2) + \frac{1}{2}(x-2) + \frac{(-1/4)}{2!}(x-2)^2 + \frac{1/4}{3!}(x-2)^3 + \frac{(-3/8)}{4!}(x-2)^4 + \dots$$
$$= \ln(2) + \frac{(x-2)}{2} - \frac{(x-2)^2}{8} + \frac{(x-2)^3}{24} - \frac{(x-2)^4}{64} + \dots \qquad [\text{Simplifying}]$$

We can visualise some of the approximations to $\ln(x)$ near $a = 2$. Next we plot the first few terms of the series found in Example 17. By using a computer algebra package or graphical calculator plot the following graphs on the same axis:

$\ln(x)$, $\ln(2) + \frac{(x-2)}{2}$ (linear approx), $\ln(2) + \frac{(x-2)}{2} - \frac{(x-2)^2}{8}$ (quadratic approx) and $\ln(2) + \frac{(x-2)}{2} - \frac{(x-2)^2}{8} + \frac{(x-2)^3}{24}$ (cubic approx).

We obtain:

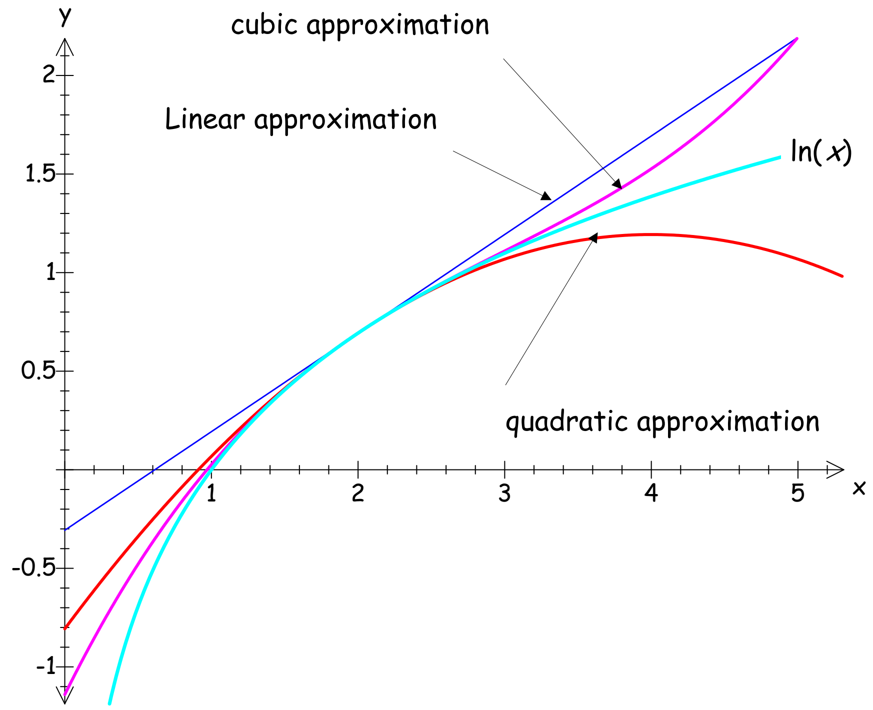

Fig 20

Note that as we increase the number of terms we get better approximations to $\ln(x)$. Again if we consider the infinite series of Example 17 then we can say that they are equal.

Example 18

Obtain the Taylor series for $\sin(x)$ about the point $a=\frac{\pi}{4}$.

Solution

Let $f(x)=\sin(x)$ then we have

$$f(x)=\sin(x) \qquad f\left(\frac{\pi}{4}\right)=\sin\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$$

$$f'(x)=\cos(x) \quad [\text{Differentiating}] \qquad f'\left(\frac{\pi}{4}\right)=\cos\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$$

$$f''(x)=-\sin(x) \quad [\text{Differentiating}] \qquad f''\left(\frac{\pi}{4}\right)=-\sin\left(\frac{\pi}{4}\right)=-\frac{1}{\sqrt{2}}$$

$$f'''(x)=-\cos(x) \quad [\text{Differentiating}] \qquad f'''\left(\frac{\pi}{4}\right)=-\cos\left(\frac{\pi}{4}\right)=-\frac{1}{\sqrt{2}}$$

$$f^{(4)}(x)=\sin(x) \quad [\text{Differentiating}] \qquad f^{(4)}\left(\frac{\pi}{4}\right)=\sin\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$$

Substituting these values into the Taylor series

(3.8) $$f(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\ldots$$

with $f(x)=\sin(x)$ and $a=\frac{\pi}{4}$ we have

$$\sin(x)=f\left(\frac{\pi}{4}\right)+f'\left(\frac{\pi}{4}\right)\left(x-\frac{\pi}{4}\right)+\frac{f''\left(\frac{\pi}{4}\right)}{2!}\left(x-\frac{\pi}{4}\right)^2+\frac{f'''\left(\frac{\pi}{4}\right)}{3!}\left(x-\frac{\pi}{4}\right)^3+\frac{f^{(4)}\left(\frac{\pi}{4}\right)}{3!}\left(x-\frac{\pi}{4}\right)^4+\ldots$$

$$=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\left(x-\frac{\pi}{4}\right)-\frac{1}{\sqrt{2}}\frac{1}{2!}\left(x-\frac{\pi}{4}\right)^2-\frac{1}{\sqrt{2}}\frac{1}{3!}\left(x-\frac{\pi}{4}\right)^3+\frac{1}{\sqrt{2}}\frac{1}{4!}\left(x-\frac{\pi}{4}\right)^4+\ldots$$

$$=\frac{1}{\sqrt{2}}\left[1+\left(x-\frac{\pi}{4}\right)-\frac{1}{2!}\left(x-\frac{\pi}{4}\right)^2-\frac{1}{3!}\left(x-\frac{\pi}{4}\right)^3+\frac{1}{4!}\left(x-\frac{\pi}{4}\right)^4+\ldots\right] \qquad \left[\begin{array}{l}\text{Taking out common}\\ \text{factor } 1/\sqrt{2}\end{array}\right]$$

Remember for the trigonometric functions the argument *x* must be in **radians**.

D2 Remainder Term

With series expansion of functions we don’t really know how good our approximation is for a finite number of terms. We need a method to find the difference between the actual value of the function at a particular point and the value of the Taylor series at that point for some given number of terms.

Consider the above example with $f(x)=\sin(x)$ at $x=\frac{\pi}{2}$.

Let's say we want to find the difference between the actual value $\sin\left(\frac{\pi}{2}\right)=1$ and the value of the first **three** terms of the Taylor expansion above:

$$\sin(x)=\frac{1}{\sqrt{2}}\left[1+\left(x-\frac{\pi}{4}\right)-\frac{1}{2!}\left(x-\frac{\pi}{4}\right)^2-\ldots\right]$$

$$\sin\left(\frac{\pi}{2}\right)\simeq\frac{1}{\sqrt{2}}\left[1+\left(\frac{\pi}{2}-\frac{\pi}{4}\right)-\frac{1}{2}\left(\frac{\pi}{2}-\frac{\pi}{4}\right)^2\right]$$

$$=\frac{1}{\sqrt{2}}\left[1+\frac{\pi}{4}-\frac{1}{2}\left(\frac{\pi}{4}\right)^2\right]=1.044377642$$

The absolute difference between the actual value $\sin\left(\frac{\pi}{2}\right)=1$ and 1.044377642 is 0.044377642.

We can show this on a graph as follows:

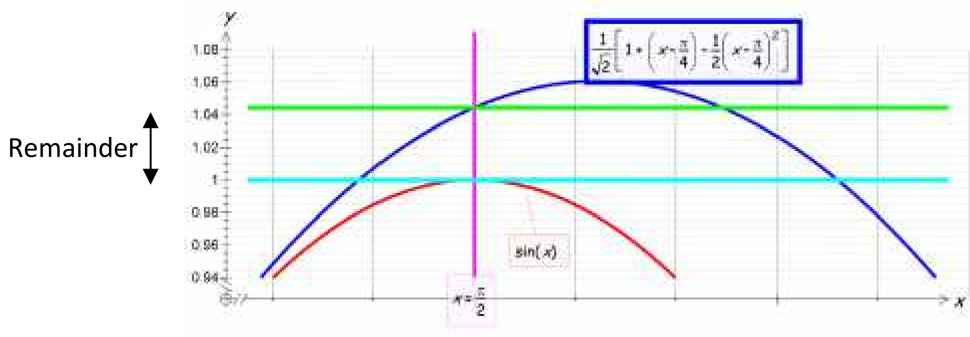

Fig 21

This difference is normally called the **remainder term** or **error term**.

We can write Taylor's Theorem as a Taylor polynomial plus the remainder term as follows:

The Taylor series for a function $f(x)$ about a point $x=a$ is given by:

(3.9) $$f(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\ldots+\frac{f^{(n)}(a)}{n!}(x-a)^n+R_n(x)$$

where the remainder term $R_n(x)=\frac{f^{n+1}(c)}{(n+1)!}(x-a)^{n+1}$ and $c\in\,]a,\quad x[$.

The given function $f(x)$ is $(n+1)$ times differentiable on an open interval containing the points *a* and *x*.

Proof.

See David Brannan – A First Course in Mathematical Analysis, pages 320 – 321.

We are interested in an estimate for the remainder term $R_n(x)$. *What does the size of the remainder term* $R_n(x)=\frac{f^{n+1}(c)}{(n+1)!}(x-a)^{n+1}$ *depend on?*

The larger the value of *n* (which means taking more terms) the smaller our remainder. *Why?*

Because in the denominator of the remainder term we have $(n+1)!$ For example if we considered the cubic expansion of $\sin(x)$ rather than the quadratic on the previous page then the remainder term would be smaller.

The following Corollary gives us a maximum absolute function for the remainder term.

Corollary (3.10). Let $f(x)$ be $(n+1)$ times differentiable on an open interval containing the points *a* and *x*. If for all $c\in\,]a,\ x[$ we have

$$\left|f^{n+1}(c)\right|\le M \quad \text{where } M \text{ is a positive number}$$

then $f(x)=T_n(x)+R_n(x)$ where $T_n(x)$ is the *n*th degree Taylor polynomial and

$$\left|R_n(x)\right|\le\frac{M}{(n+1)!}\left|x-a\right|^{n+1}$$

This $\left|R_n(x)\right|\le\frac{M}{(n+1)!}\left|x-a\right|^{n+1}$ is called **Lagrange's form of the remainder term**. It is similar to the other Taylor series terms but $f^{n+1}(c)$ is evaluated at *c* rather than *a*.

Proof.

By Taylor's Theorem (3.9) on the previous page we have

$$R_n(x)=\frac{f^{n+1}(c)}{(n+1)!}(x-a)^{n+1}$$

We are given that $\left|f^{n+1}(c)\right|\le M$ for all $c\in\,]a,\ x[$ therefore

$$\left|R_n(x)\right|=\frac{\left|f^{n+1}(c)\right|}{(n+1)!}\left|x-a\right|^{n+1}\le\frac{M}{(n+1)!}\left|x-a\right|^{n+1}$$

□

Example 19

By applying Taylor's Theorem to the function $f(x)=\cos(x)$ with $a=\frac{\pi}{2}$, show that

$$\cos(x)=-\left(x-\frac{\pi}{2}\right)+\frac{1}{3!}\left(x-\frac{\pi}{2}\right)^3+R_3(x)$$

where $\left|R_3(x)\right|\le\frac{1}{24}\left|x-\frac{\pi}{2}\right|^4$.

Solution

Let $f(x)=\cos(x)$ then the Taylor series expansion about the point $a=\frac{\pi}{2}$ is given by:

$$f(x)=\cos(x),\qquad f\left(\frac{\pi}{2}\right)=\cos\left(\frac{\pi}{2}\right)=0$$

$$f'(x)=-\sin(x),\qquad f'\left(\frac{\pi}{2}\right)=-\sin\left(\frac{\pi}{2}\right)=-1$$

$$f''(x)=-\cos(x),\qquad f''\left(\frac{\pi}{2}\right)=-\cos\left(\frac{\pi}{2}\right)=0$$

$$f'''(x)=\sin(x),\qquad f'''\left(\frac{\pi}{2}\right)=\sin\left(\frac{\pi}{2}\right)=1$$

Substituting these into the Taylor polynomial of degree 3 for $f(x)=\cos(x)$ gives:

$$\begin{aligned}T_3(x)&=f\left(\frac{\pi}{2}\right)+f'\left(\frac{\pi}{2}\right)\left(x-\frac{\pi}{2}\right)+\frac{f''\left(\frac{\pi}{2}\right)}{2!}\left(x-\frac{\pi}{2}\right)^2+\frac{f'''\left(\frac{\pi}{2}\right)}{3!}\left(x-\frac{\pi}{2}\right)^3\\&=0-1\left(x-\frac{\pi}{2}\right)+\frac{0}{2!}\left(x-\frac{\pi}{2}\right)^2+\frac{1}{3!}\left(x-\frac{\pi}{2}\right)^3\\&=-\left(x-\frac{\pi}{2}\right)+\frac{1}{3!}\left(x-\frac{\pi}{2}\right)^3\end{aligned}$$

The remainder term $R_n(x)=\frac{f^{n+1}(c)}{(n+1)!}(x-a)^{n+1}$ for $n=3$ is:

$$R_3(x)=\frac{f^{(3+1)}(c)}{(3+1)!}(x-a)^{3+1}=\frac{f^{(4)}(c)}{4!}(x-a)^4$$

where $c \in \mathbb{R}$. By the above Corollary we have $\left|R_n(x)\right| \le \frac{M}{(n+1)!}\left|x-a\right|^{n+1}$ where $\left|f^{n+1}(c)\right| \le M$.

We need to evaluate $f^{(4)}(c)$:

$$f^{(4)}(c) = \cos(c)$$

The cosine function lies between -1 and +1 and taking the absolute value gives

$$\left|f^{(4)}(c)\right| = \left|\cos(c)\right| \le 1 = M$$

Hence $\left|R_3(x)\right| \le \frac{1}{4!}\left|x-\frac{\pi}{2}\right|^4 = \frac{1}{24}\left|x-\frac{\pi}{2}\right|^4$.

In the above example the remainder (error) at $x = 2$ is

$$\left|R_3(2)\right| \le \frac{1}{24}\left|2-\frac{\pi}{2}\right|^4 = 0.001414 = 1.414\times 10^{-3}$$

This means the remainder (error) is less than 1.5 parts in a thousand at $x = 2$ for the cubic approximation to $\cos(x)$.

Example 20

By applying Taylor's Theorem to the function $f(x) = e^x$ with $a = 0$, show that

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + R_4(x)$$

where $\left|R_4(1)\right| \le \frac{e}{120}$.

<u>Solution</u>

Let $f(x) = e^x$ then

$$f(x) = f'(x) = f''(x) = f'''(x) = f^{(4)}(x) = f^{(5)}(x) = e^x$$

In each case we have $f(0) = f'(0) = f''(0) = f'''(0) = f^{(4)}(0) = f^{(5)}(0) = e^0 = 1$

Substituting these into the Taylor polynomial of degree 4 for $f(x) = e^x$ with $a = 0$ is:

$$T_4(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \frac{f^{(4)}(0)}{4!}(x-0)^4$$

$$= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4$$

The remainder term $R_n(x) = \frac{f^{n+1}(c)}{(n+1)!}(x-a)^{n+1}$ for $n = 4$. *What values does c take?*

By the above Corollary (3.10) we have $c \in \left]a, \quad x\right[$. *What are the a and x equal to in this case?*

We are given $a = 0$ and because we want to find a maximum value for $\left|R_4(1)\right|$ therefore $x = 1$.

Hence $c \in \left]0, \quad 1\right[$ and the remainder term is given by:

$$R_4(x) = \frac{f^{(4+1)}(c)}{(4+1)!}(x-0)^{4+1} = \frac{f^{(5)}(c)}{5!}x^5$$

We have $\left|R_n(x)\right| \leq \frac{M}{(n+1)!}\left|x-a\right|^{n+1}$ where $\left|f^{n+1}(c)\right| \leq M$.

We need to evaluate $f^{(5)}(c)$:

$$f^{(5)}(c) = e^c$$

Since $f(x) = e^x$ is an increasing function and $c \in \left]0, \quad 1\right[$ so

$$\left|f^{(5)}(c)\right| = \left|e^c\right| \leq e^1 = e$$

Hence $\left|R_4(1)\right| = \frac{\left|f^{(5)}(c)\right|}{5!}\left|1\right|^5 \leq \frac{e}{120}$.

This means the remainder term $\left|R_4(1)\right| \leq \frac{e}{120}$ is less than 3 parts in 120.

Example 21

Calculate the Taylor polynomial $T_3(x)$ to the function $f(x) = \frac{1}{x+2}$ at $a = 1$.

Show that $T_3(x)$ approximates $f(x)$ with an error less than 5×10^{-3} on the interval $\left[1, \quad 2\right]$.

Solution

Let $f(x)=\dfrac{1}{x+2}=(x+2)^{-1}$ then

$$f(x)=\frac{1}{x+2}, \qquad f(1)=\frac{1}{1+2}=\frac{1}{3}$$

$$f'(x)=-(x+2)^{-2}, \qquad f'(1)=-\frac{1}{(1+2)^2}=-\frac{1}{9}$$

$$f''(x)=2(x+2)^{-3}, \qquad f''(1)=\frac{2}{(1+2)^3}=\frac{2}{27}$$

$$f'''(x)=-6(x+2)^{-4}, \qquad f'''(1)=\frac{-6}{(1+2)^4}=-\frac{6}{81}=-\frac{2}{27}$$

Substituting these into the Taylor polynomial of degree 3 for $f(x)=\dfrac{1}{x+2}$ at $a=1$ is:

$$\begin{aligned}T_3(x)&=f(1)+f'(1)(x-1)+\frac{f''(1)}{2!}(x-1)^2+\frac{f'''(1)}{3!}(x-1)^3\\&=\frac{1}{3}-\frac{1}{9}(x-1)+\frac{2}{27\times 2!}(x-1)^2-\frac{2}{27\times 3!}(x-1)^3\\&=\frac{1}{3}-\frac{1}{9}(x-1)+\frac{1}{27}(x-1)^2-\frac{1}{81}(x-1)^3\end{aligned}$$

The remainder term $R_3(x)=\dfrac{f^{(3+1)}(c)}{(3+1)!}(x-1)^{3+1}=\dfrac{f^{(4)}(c)}{4!}(x-1)^4$ where $c\in[1,\quad 2]$.

From above we have $f'''(x)=-6(x+2)^{-4}$ so

$$f^{(4)}(x)=24(x+2)^{-5}, \qquad f^{(4)}(c)=\frac{24}{(2+c)^5}$$

Since $c\in[1,\quad 2]$ which means that $1\le c\le 2$ and in $f^{(4)}(c)$ the *c* is in the denominator, we have

$$\left|f^{(4)}(c)\right|=\frac{24}{(2+c)^5}\le\frac{24}{(2+1)^5}=\frac{24}{3^5}=\frac{8}{81}=M$$

From the Corollary we have $\left|R_3(x)\right|\le\dfrac{M}{(3+1)!}\left|x-1\right|^{3+1}$. Since the interval is $[1,\quad 2]$ so the largest value of *x* is 2. Hence

$$\left|R_3(x)\right| \le \frac{8}{81}\frac{1}{(3+1)!}\left|2-1\right|^{3+1} = \frac{8}{81}\times\frac{1}{24} = \frac{1}{243} \simeq 0.0041$$

Hence the error is less than 5×10^{-3} on the interval $[1, \quad 2]$.

The remainder or error term is less than 5 parts in a 1000 from the actual value for a cubic Taylor approximation to $f(x) = \dfrac{1}{x+2}$.

SUMMARY

A series expansion about a real number $x = a$ where *a* may not be zero is called a Taylor series.

The remainder term $R_n(x)$ in a Taylor series is $\left|R_n(x)\right| \le \dfrac{\left|f^{n+1}(c)\right|}{(n+1)!}\left|x-a\right|^{n+1}$ where $c \in \left]a, \quad x\right[$.