Orthogonal Diagonalization Page 537

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} \qquad D^{-1} = \begin{pmatrix} 1 & & & \\ & 1/2 & & 0 \\ & 0 & 1/3 & \\ & & & 1/4 \end{pmatrix}$$

$$D^{10} = \begin{pmatrix} 1^{10} & & & \\ & 2^{10} & & 0 \\ & 0 & 3^{10} & \\ & & & 4^{10} \end{pmatrix}$$

Proof: $A^T = A$

By (7.24) we have

$$Q^{-1} A Q = D.$$

Pre-multiply by Q:

$$Q(Q^{-1} A Q) = QD$$

$$\underbrace{(Q Q^{-1})}_{=I} A Q = QD$$

$$IAQ = QD$$

$$AQ = QD.$$

Post $\times$ by Q^{-1}:

$$A\underbrace{(Q Q^{-1})}_{=I} = QDQ^{-1}$$

$$A = QDQ^{-1}$$

$$\underline{A = Q D Q^T}$$

$$A = Q D Q^T$$

Transpox b/s:

$$A^T = (Q D Q^T)^T \quad \boxed{(ABC)^T = C^T B^T A^T}$$

$$= (Q^T)^T D^T Q^T$$

$$= Q D Q^T = A$$

$$\underline{A^T = A}$$

Sol4:

$$\lambda_1 = -1, \quad \underline{u} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

$$\lambda_2 = 4, \quad \underline{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

$$Q^* = \begin{pmatrix} \underline{u} & \underline{v} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$$

$$\hat{\underline{u}} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$\hat{\underline{v}} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$Q = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}.$$

$$Q^T A Q = D = \begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix}.$$

Sol^n:

$$\underline{u} \cdot \underline{v} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} = -4 \neq 0$$

GS:

$$p_1 = u = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

$$p_2 = v - \frac{\langle p_1, v \rangle}{\|p_1\|^2} \underline{p_1}$$

$$= \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} - \frac{\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}}{\left\| \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \right\|^2} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} - \frac{-4}{5} \cancel{\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} + \frac{4}{5} \cancel{\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -2 + 8/5 \\ 0 + 4/5 \\ 1 + 0 \end{pmatrix}$$

$$= \begin{pmatrix} -2/5 \\ 4/5 \\ 1 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} -2 \\ 4 \\ 5 \end{pmatrix} = \underline{p_2}$$

Let $\underline{p}_2^* = \begin{pmatrix} -2 \\ 4 \\ 5 \end{pmatrix}$

$$\underline{p}_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \quad \underline{p}_2^* = \begin{pmatrix} -2 \\ 4 \\ 5 \end{pmatrix} \ \& \ \underline{w} = \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix}$$

$$Q^+ = \begin{pmatrix} \underline{p}_1 & \underline{p}_2^* & \underline{w} \end{pmatrix}$$

Normalising

$$\underline{\hat{p}}_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

$$\underline{\hat{p}}_2^* = \frac{1}{\sqrt{4+16+25}} \begin{pmatrix} -2 \\ 4 \\ 5 \end{pmatrix} = \frac{1}{\sqrt{45}} \begin{pmatrix} -2 \\ 4 \\ 5 \end{pmatrix}$$

$$\underline{\hat{w}} = \frac{1}{3} \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix}$$

$$Q = \begin{pmatrix} \underline{\hat{p}}_1 & \underline{\hat{p}}_2^* & \underline{\hat{w}} \end{pmatrix}$$

$$= \begin{pmatrix} 2/\sqrt{5} & -2/\sqrt{45} & -1/3 \\ 1/\sqrt{5} & 4/\sqrt{45} & 2/3 \\ 0 & \frac{5}{\sqrt{45}} & -2/3 \end{pmatrix}.$$

$$QD = AQ = D = \begin{pmatrix} 3 & & 0 \\ & 3 & \\ 0 & & -6 \end{pmatrix}.$$