SECTION E **Functions of a General Period**

By the end of this section you will be able to

- obtain a Fourier series of a function of any period

E1 **Fourier Coefficients of a General Period**

Until now the functions we have discussed so far have *all* had a period of 2π. Of course there are many more functions whose period is *not* equal to 2π. For example

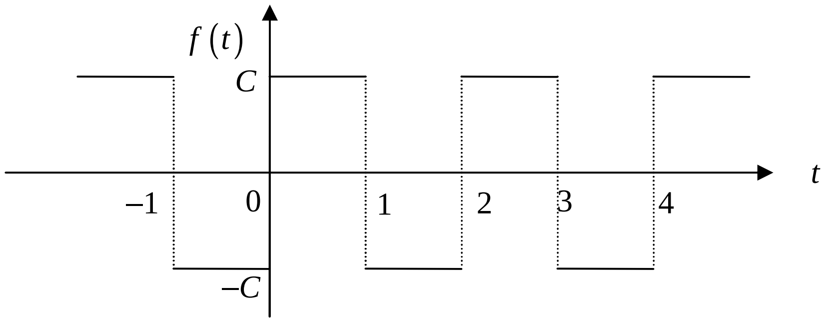

Figure 38

What is the period of $f(t)$*?*

Period = 2

We can convert our formulae for the Fourier coefficients given by

(17.3) $$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,\mathrm{d}t$$

(17.4) $$A_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(kt)\,\mathrm{d}t$$

(17.5) $$B_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(kt)\,\mathrm{d}t$$

to cover a general period $2L$. Note that formulae (17.3) to (17.5) are *only* valid for functions with period $= 2\pi$. The Fourier coefficients for a function with period $= 2L$ are given by

(17.17) $$A_0 = \frac{1}{2L}\int_{-L}^{L} f(t)\,\mathrm{d}t$$

(17.18) $$A_k = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{k\pi t}{L}\right)\mathrm{d}t$$

(17.19) $$B_k = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{k\pi t}{L}\right)\mathrm{d}t$$

Note that substituting $L = \pi$ into these gives us the Fourier coefficients of the Fourier series of $f(t)$ with period 2π as they give us formulae (17.3, 4 and 5).

The Fourier series of a periodic function with period $2L$ is given by

(17.20) $$f(t) = A_0 + A_1 \cos\left(\frac{\pi t}{L}\right) + A_2 \cos\left(\frac{2\pi t}{L}\right) + \cdots + B_1 \sin\left(\frac{\pi t}{L}\right) + B_2 \sin\left(\frac{2\pi t}{L}\right) + \cdots$$

Let's do an example so that you can become familiar with these formulae.

Example 11

Determine the Fourier coefficients for the following periods:

(a) $2L = 10$ (b) $2L = l$ (c) $2L = \frac{2\pi}{\omega}$

Solution

(a) Substituting $L = 10/2 = 5$ into

(17.17) $$A_0 = \frac{1}{2L}\int_{-L}^{L} f(t)\,\mathrm{d}t$$

gives

$$A_0 = \frac{1}{10}\int_{-5}^{5} f(t)\,\mathrm{d}t$$

Similarly for A_k we substitute $L = 5$ into

(17.18) $$A_k = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{k\pi t}{L}\right)\mathrm{d}t$$

which gives

$$A_k = \frac{1}{5}\int_{-5}^{5} f(t)\cos\left(\frac{k\pi t}{5}\right)\mathrm{d}t$$

Similarly we have

$$B_k = \frac{1}{5}\int_{-5}^{5} f(t)\sin\left(\frac{k\pi t}{5}\right)\mathrm{d}t$$

(b) Substituting $L = l/2$ into (17.17) to (17.19) gives

$$A_0 = \frac{1}{2l/2}\int_{-l/2}^{l/2} f(t)\,\mathrm{d}t = \frac{1}{l}\int_{-l/2}^{l/2} f(t)\,\mathrm{d}t \qquad \left[\text{Cancelling 2's}\right]$$

$$A_k = \frac{1}{l/2}\int_{-l/2}^{l/2}\left[f(t)\cos\left(\frac{k\pi t}{l/2}\right)\right]\mathrm{d}t$$

$$= \frac{2}{l}\int_{-l/2}^{l/2}\left[f(t)\cos\left(\frac{2k\pi t}{l}\right)\right]\mathrm{d}t$$

$$B_k = \frac{1}{l/2}\int_{-l/2}^{l/2}\left[f(t)\sin\left(\frac{k\pi t}{l/2}\right)\right]\mathrm{d}t$$

$$= \frac{2}{l}\int_{-l/2}^{l/2}\left[f(t)\sin\left(\frac{2k\pi t}{l}\right)\right]\mathrm{d}t$$

(c) Substituting $L = \dfrac{\pi}{\omega}$ into

(17.17) $$A_0 = \frac{1}{2L}\int_{-L}^{L} f(t)\,\mathrm{d}t$$

gives

$$A_0 = \frac{1}{2\pi/\omega}\left[\int_{-\pi/\omega}^{\pi/\omega} f(t)\,\mathrm{d}t\right] = \frac{\omega}{2\pi}\left[\int_{-\pi/\omega}^{\pi/\omega} f(t)\,\mathrm{d}t\right] \qquad \left[\text{Because } \frac{1}{2\pi/\omega} = \frac{\omega}{2\pi}\right]$$

The Cosine coefficients are given by:

$$A_k = \frac{1}{\pi/\omega}\left[\int_{-\pi/\omega}^{\pi/\omega} f(t)\cos\left(\frac{k\pi t}{\pi/\omega}\right)\mathrm{d}t\right]$$
$$= \frac{\omega}{\pi}\left[\int_{-\pi/\omega}^{\pi/\omega} f(t)\cos(k\omega t)\,\mathrm{d}t\right] \qquad \left[\text{Because } \frac{k\pi t}{\pi/\omega} = k\pi t \div \frac{\pi}{\omega} = k\not{\pi} t \times \frac{\omega}{\not{\pi}}\right]$$

Sine coefficients:

$$B_k = \frac{1}{\pi/\omega}\left[\int_{-\pi/\omega}^{\pi/\omega} f(t)\sin\left(\frac{k\pi t}{\pi/\omega}\right)\mathrm{d}t\right]$$
$$= \frac{\omega}{\pi}\left[\int_{-\pi/\omega}^{\pi/\omega} f(t)\sin(k\omega t)\,\mathrm{d}t\right] \qquad \left[\text{Because } \frac{k\pi t}{\pi/\omega} = k\pi t \div \frac{\pi}{\omega} = k\not{\pi} t \times \frac{\omega}{\not{\pi}}\right]$$

What are the Fourier coefficients for odd and even functions with period = 2L?

We integrate between 0 to L (remember 0 to π in the case when the period is 2π.)

Hence the Fourier coefficients are given by

(17.21) $$A_0 = \frac{1}{L}\int_0^L f(t)\,\mathrm{d}t$$

(17.22) $$A_k = \frac{2}{L}\int_0^L f(t)\cos\left(\frac{k\pi t}{L}\right)\mathrm{d}t$$

(17.23) $$B_k = \frac{2}{L}\int_0^L f(t)\sin\left(\frac{k\pi t}{L}\right)\mathrm{d}t$$

Example 12

Determine the Fourier series of the rectangular waveform $f(t)$ shown below:

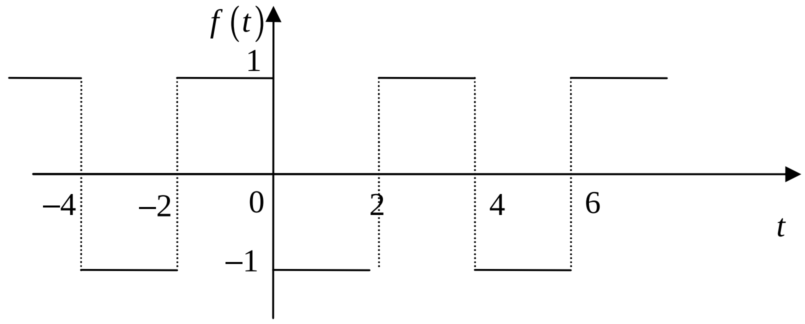

Figure 39

Solution

What is the period of the given waveform $f(t)$*?*

Period $2L = 4$. *What type of waveform is* $f(t)$*?*

The given $f(t)$ is an *odd* function therefore there are *no* constant and cosine terms:

$$A_0 = A_k = 0$$

We only have sine terms and the coefficients are given by

(17.23) $$B_k = \frac{2}{L}\int_0^L f(t)\sin\left(\frac{k\pi t}{L}\right)\mathrm{d}t$$

In our case we have $L = 2$. *What is* $f(t)$ *equal to between 0 and 2?*

By the graph of Fig. 39 we have $f(t) = -1$ for t between 0 and 2. Therefore

$$B_k = \frac{2}{2}\int_0^2 (-1)\sin\left(\frac{k\pi t}{2}\right)\mathrm{d}t \qquad \left[\begin{array}{l}\text{Substituting } f(t) = -1 \\ \text{and } L = 2 \text{ into (17.22)}\end{array}\right]$$

$$= -\int_0^2 \sin\left(\frac{k\pi t}{2}\right)\mathrm{d}t$$

$$= -\left[-\frac{\cos(k\pi t/2)}{k\pi/2}\right]_0^2 \qquad \left[\text{Integrating by } \int \sin(mt)\,\mathrm{d}t = -\frac{\cos(mt)}{m}\right]$$

$$= \frac{2}{k\pi}\left[\cos\left(\frac{k\pi t}{2}\right)\right]_0^2 \qquad \left[\begin{array}{l}\text{Taking out the common} \\ \text{factor } -\dfrac{1}{k\pi/2} = -\dfrac{2}{k\pi}\end{array}\right]$$

$$= \frac{2}{k\pi}\left[\cos(k\pi) - \cos(0)\right] \qquad \left[\begin{array}{l}\text{Subtituting limits} \\ t = 2 \text{ and } t = 0\end{array}\right]$$

$$B_k = \frac{2}{k\pi}\left[\cos(k\pi) - 1\right] \qquad (\dagger)$$

Using our well established result:

$$\cos(k\pi) = \begin{cases} 1 & \text{if } k = \text{even} \\ -1 & \text{if } k = \text{odd}\end{cases}$$

If k is even then substituting $\cos(k\pi) = 1$ into (†) yields

$$B_k = \frac{2}{k\pi}\left[\cos\left(k\pi\right) - 1\right] = \frac{2}{k\pi}\underbrace{\left[1 - 1\right]}_{=0} = 0$$

If k is odd then substituting $\cos\left(k\pi\right) = -1$ into (†) gives

$$B_k = \frac{2}{k\pi}\left[\cos\left(k\pi\right) - 1\right] = \frac{2}{k\pi}\left[-1 - 1\right] = \frac{2}{k\pi}\left(-2\right) = -\frac{4}{k\pi}$$

Putting $A_0 = 0$, $A_k = 0$, $B_k = 0$ [for even k] and $B_k = -\frac{4}{k\pi}$ [for odd k] into the generic Fourier series

(17.20) $$f\left(t\right) = A_0 + A_1\cos\left(\frac{\pi t}{L}\right) + A_2\cos\left(\frac{2\pi t}{L}\right) + \cdots + B_1\sin\left(\frac{\pi t}{L}\right) + B_2\sin\left(\frac{2\pi t}{L}\right) + \cdots$$

Gives (recall $L = 2$)

$$f\left(t\right) = \underbrace{0}_{\text{Constant term}} + \underbrace{0}_{\text{Cosine terms}} \underbrace{-\frac{4}{\pi}\sin\left(\frac{\pi t}{2}\right) + 0 - \frac{4}{3\pi}\sin\left(\frac{3\pi t}{2}\right) + 0 - \frac{4}{5\pi}\sin\left(\frac{5\pi t}{2}\right) + 0 - \ldots}_{\text{Sine terms}}$$

$$= -\frac{4}{\pi}\left[\sin\left(\frac{\pi t}{2}\right) + \frac{\sin\left(3\pi t/2\right)}{3} + \frac{\sin\left(5\pi t/2\right)}{5} + \ldots\right] \qquad \left[\text{Taking out } -\frac{4}{\pi}\right]$$

What series do we obtain if we substitute $t = 1$ into this Fourier series

$$f\left(t\right) = -\frac{4}{\pi}\left[\sin\left(\frac{\pi t}{2}\right) + \frac{\sin\left(3\pi t/2\right)}{3} + \frac{\sin\left(5\pi t/2\right)}{5} + \ldots\right]?$$

By the graph of Fig. 39 at $t = 1$ we have $f\left(1\right) = -1$. Substituting $t = 1$ into the right hand side of the above Fourier series gives

$$-1 = -\frac{4}{\pi}\left[\sin\left(\frac{\pi}{2}\right) + \frac{\sin\left(3\pi/2\right)}{3} + \frac{\sin\left(5\pi/2\right)}{5} + \frac{\sin\left(7\pi/2\right)}{7} + \ldots\right]$$

$$-1 = -\frac{4}{\pi}\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\right]$$

Multiplying both sides of the last evaluation by -1 gives

$$1 = \frac{4}{\pi}\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\right]$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$

We obtain the series $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$.

SUMMARY

We can still find the Fourier series of a periodic function $f\left(t\right)$ (provided it satisfies the Dirichlet conditions) even if it does not have a period 2π. In this section we have evaluated the Fourier series of waveforms of a general period $= 2L$.