SECTION B Subsets

By the end of this section you will be able to

- carry out various operations of subsets
- understand what is meant by power set and cardinality
- prove properties of subsets

B1 Introduction to Subsets

*What do you think the term **subset** means?*

The prefix 'sub' normally means contained within a system or structure. Thus subset is a set contained within another set. The Venn diagram of the set A being a subset of the set B is given by:

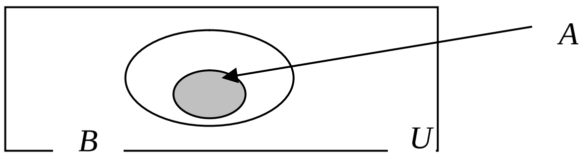

Figure 14 A is a subset of B

What is the definition of a subset?

All the elements of one set are contained within another set. In general let A and B be two given sets. If every element in set A is also in set B then we say A is a **subset** of B. We also say A is **contained** in B.

How do we denote a subset in mathematical notation?

We denote A is a subset of B by $A \subseteq B$.

An example of a subset is $\{\text{students}\} \subseteq \{\text{human race}\}$. *What set is B the subset of in Figure 14?*

The universal set U, that is $B \subseteq U$. Note that set A is also a subset of universal set U. *Is there a set which is the subset of A?*

Yes the empty set $\varnothing$ is a subset of A, that is $\varnothing \subseteq A$. The empty set $\varnothing$ is a subset of **every** set. Also note that A is a subset of itself, that is $A \subseteq A$.

Can you think of any other examples of subsets?

There are an infinite number of examples which you could easily make up such as the following:

1. Let $A = \{\text{Marilyn Munroe, Liz Hurley, Michelle Pfeiffer}\}$ and $B = \{\text{women}\}$ then A is a subset of B which we denote by $A \subseteq B$.
2. Let $A = \{\text{Prime Ministers of Britian}\}$ and $B = \{\text{Thatcher, Major, Blair}\}$ then $B \subseteq A$.
3. Let $A = \{x \mid x \text{ is an even number}\}$ and $B = \{2, 4, 6, 8\}$. Clearly $B \subseteq A$.
4. Let $A = \{x \in \mathbb{R} \mid -2 \le x \le 2\}$ and $B = \{x \in \mathbb{R} \mid -4 \le x \le 4\}$ as shown in the diagram below:

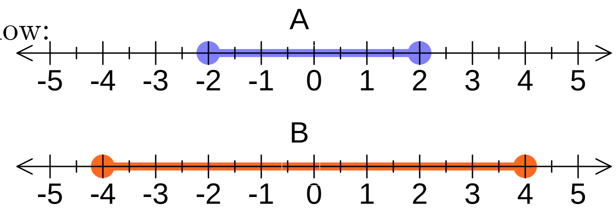

Figure 15

We have A is a subset of B which is denoted by $A \subseteq B$.

5. Let $A = \{a, e, i, o, u\}$ and $B = \{\text{Letters of the alphabet}\}$. Again we have $A \subseteq B$.

There are many more interesting examples you can create.

The formal definition of subset is:

Definition (I.7). Given sets A and B. The set A is a **subset** of B if and only if **every** element of the set A is also in the set B and is denoted by $A \subseteq B$.

In some mathematics books the author may make the distinction between **proper subset** and **subset**. *What do you think is the difference between **proper subset** and **subset**?*

A set A is a **proper subset** of a set B if A is subset of B but $A \neq B$ as shown in Figure 14.

The set A is a **subset** of B if A is a subset of B but the set A may equal B. For example A is a subset of A but is **not** a proper subset of A. In notation terms we will **not** make the distinction between proper subset or subset. In **both** cases we will use the notation $A \subseteq B$ to mean proper subset and subset.

*What notation do we use when set A is **not** a subset of the set B?*

$$A \not\subseteq B \quad [A \text{ is } \textbf{not} \text{ a subset of } B]$$

You may also have $A \subseteq B \subseteq C$. *What does this notation, $A \subseteq B \subseteq C$, mean?*

Set A is a subset of set B and B is a subset of C. The Venn diagram of this is:

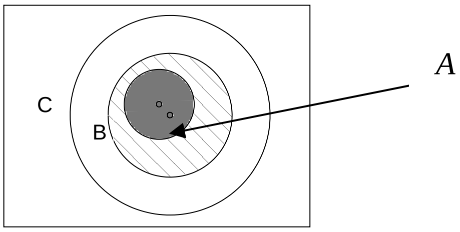

Figure 16

Example 11

Determine whether the set A in the following is a subset of the set B.

(a) $A = \mathbb{N}$ and $B = \mathbb{Z}$ (b) $A = \mathbb{Z}$ and $B = \mathbb{Q}$ (c) $A = \mathbb{Q}$ and $B = \mathbb{R}$

(d) $A = \mathbb{R}$ and $B = \mathbb{C}$

Solution.

To answer each of these we need to know what the symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ represent. *What do these symbols signify?*

From section A we have:

$\mathbb{N} =$ Natural numbers. These are positive whole numbers.

$\mathbb{Z} =$ Integers. These are positive and negative whole numbers and zero.

$\mathbb{Q} =$ Rational Numbers. These are ratios of integers and includes integers.

$\mathbb{R} =$ Real numbers. These are rational as well as irrational numbers.

$\mathbb{C} =$ Complex numbers. These are numbers of the form $a + b\sqrt{-1}$ and include all the real numbers.

(a) The set of positive whole numbers $\mathbb{N}$ is a subset of the set of integers $\mathbb{Z}$ so we have $\mathbb{N} \subseteq \mathbb{Z}$.

(b) *Is* $\mathbb{Z} \subseteq \mathbb{Q}$ *or* $\mathbb{Q} \subseteq \mathbb{Z}$ *?*

Since the integers, $\mathbb{Z}$, can be written as fractions such as $3 = \frac{3}{1}$ therefore the integers are a subset of the rational numbers $\mathbb{Q}$. Thus $\mathbb{Z} \subseteq \mathbb{Q}$.

(c) *Is* $\mathbb{Q}$ *a subset of* $\mathbb{R}$ *?*

Since the real numbers, $\mathbb{R}$, contains the rational numbers, $\mathbb{Q}$, and the irrational numbers therefore $\mathbb{Q}$ is subset of $\mathbb{R}$, that is $\mathbb{Q} \subseteq \mathbb{R}$.

(d) Remember the complex numbers, $\mathbb{C}$, contains all the real numbers, $\mathbb{R}$, therefore $\mathbb{R} \subseteq \mathbb{C}$.

In each of the above cases we have $A \subseteq B$.

Note that for the above example we have $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$. We can remember this by the mnemonic "Nine Zulu Queens Rule China". The Venn diagram of this is:

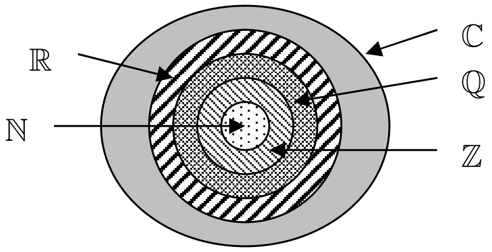

Figure 17

Example 12

Determine which of these sets A, B, C and D are a subset of the other:

$$A = \{x \in \mathbb{Z} \mid -5 \le x \le 5\}, \; B = \{x \in \mathbb{Z} \mid x^2 - 1 = 0\}$$

$$C = \{x \in \mathbb{R} \mid 2x^2 - x = 0\} \text{ and } D = \{x \in \mathbb{Q} \mid 0 \le x \le 1\}$$

Solution.

First we write out the members of each of these sets:

$$A = \{x \in \mathbb{Z} \mid -5 \le x \le 5\} = \{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$$

How do we evaluate $B = \{x \in \mathbb{Z} \mid x^2 - 1 = 0\}$ *?*

Solve the quadratic equation $x^2 - 1 = 0$:

$$x^2 - 1 = 0$$

$$(x-1)(x+1) = 0 \qquad [\text{Factorising}]$$

$$x = 1, \; x = -1$$

Thus $B = \{-1, 1\}$. *Is there any relationship between the sets A and B?*

Clearly the numbers -1 and 1 are in the set

$A = \{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$ therefore the set B is a subset of A, which we denote by $B \subseteq A$.

How do we determine the set $C = \{x \in \mathbb{R} \mid 2x^2 - x = 0\}$ *?*

Again solve the quadratic equation $2x^2 - x = 0$:

$$2x^2 - x = 0$$

$$x(2x-1) = 0 \qquad \text{gives} \quad x = 0, \quad x = \frac{1}{2}$$

Hence the set $C = \left\{0, \frac{1}{2}\right\}$.

What is the set $D = \{x \in \mathbb{Q} \mid 0 \le x \le 1\}$ *equal to?*

It is the set of all the rational numbers between 0 and 1. Numbers such as

$$0,\ \frac{1}{2},\ \frac{2}{3},\ \frac{9}{10}\ \text{etc}$$

Since the set D contains the members of the set $C=\left\{0,\ \frac{1}{2}\right\}$ therefore $C\subseteq D$.

In summary we have $B\subseteq A$ and $C\subseteq D$.

B2 Power Sets

Let A be any set. All the subsets of A is called the **power set** of A. For example let

$$A=\{1,\ 2,\ 3\}$$

What are the subsets of A?

Subsets of A are $\{1\}$, $\{2\}$, $\{3\}$, $\{1,\ 2\}$, $\{1,\ 3\}$ and $\{2,\ 3\}$. *Are there any other subsets of A?*

Yes the empty set $\varnothing$ and the set $A=\{1,\ 2,\ 3\}$ itself. Thus the power set of A is

$$\varnothing,\ \{1\},\ \{2\},\ \{3\},\ \{1,\ 2\},\ \{1,\ 3\},\ \{2,\ 3\}\ \text{and}\ A$$

The **power set** is also a set and it is denoted by $P(A)$ meaning the power set of A. Thus we have

$$P(A)=\{\varnothing,\ \{1\},\ \{2\},\ \{3\},\ \{1,\ 2\},\ \{1,\ 3\},\ \{2,\ 3\},\ A\}$$

Definition (I.8). Given a set A the **power set** of A is denoted $P(A)$ and is defined as all the sets included in A.

Example 13

Determine the power set of $A=\{1,\ 2,\ 3,\ 4\}$.

Solution.

Let $P(A)$ be the power set then we have

$$P(A)=\left\{\begin{matrix}\varnothing,\ \{1\},\ \{2\},\ \{3\},\ \{4\}, \\ \{1,\ 2\},\ \{1,\ 3\},\ \{1,\ 4\},\{2,\ 3\},\ \{2,\ 4\},\{3,\ 4\}, \\ \{1,\ 2,\ 3\},\ \{1,\ 2,\ 4\},\ \{1,\ 3,\ 4\},\ \{2,\ 3,\ 4\},\ A\end{matrix}\right\}$$

Note that $P(A)$ has 16 members.

The number of members in a set is called the **cardinality** of the set.

Definition (I.9). Given a set A the **cardinality** of A is denoted by $|A|$ and is defined as the number of elements of the set A.

If $A=\{a, b, c, d, f, s, z\}$ *then what is* $|A|$ *equal to?*

$$|A|=7 \qquad [\text{Because the set } A \text{ has 7 members}]$$

In general if the set A has n members then the power set $P(A)$ has 2^n members.

In Example 13 above the set A has 4 members therefore $P(A)=2^4=16$ members.

B3 Properties of Subsets

We have **not proven** any mathematical results so far but proof is at the heart of mathematics. We will provide some proofs in this section and you should be able to at least appreciate a proof at this point. However you will need to learn how to do proofs because proofs are sprinkled throughout the book. Additionally there is a very thorough discussion of proofs later on in this chapter.

This is difficult subsection because of the requirement to prove means that you need to know the definitions in minute detail.

Statement (I.10). If A is a subset of B and B is a subset of C then A is a subset of C. In notation form this means that if $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.

There is no point in just making a statement, we need to be able to prove this result. *How do we prove this result?*

We use the definition of a subset given above in (I.7) which says:

The set A is a **subset** of B if and only if **every** element of the set A is also in the set B.

We are told that $A \subseteq B$ and $B \subseteq C$ so we can use this information to deduce $A \subseteq C$.

First we let an arbitrary element x be a member of the set A and if we show that x is also in the set C then we have proven that $A \subseteq C$.

Proof.

Let x be a member of the set A. We have $A \subseteq B$ which means that every member of the set A is also in the set B therefore $x \in B$. Similarly we have $B \subseteq C$ which means that every member in set B is also in set C. Since $x \in B$ therefore it is a member of the set C, that is $x \in C$.

Thus by the above definition (I.7) we conclude that A is a subset of C, that is $A \subseteq C$.

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The filled square ■ is used to represent the end or completion of proof. This symbol is used throughout the book to signify the end of proof.

Note that proofs are difficult because you will need to know the definition of a subset thoroughly to prove the above result.

Example 14

Given sets A and B prove that

$$(A \setminus B) \subseteq A$$

How do we prove this result?

Let x be an arbitrary element of the set on the Left Hand Side, $A \setminus B$, then show that this arbitrary element is also in set A. *Why choose an arbitrary element?*

So that the element is **not** biased or planned in any way. If it works for an arbitrary element that means it will work for any element.

Proof.

Let x be an arbitrary element of the set $A \setminus B$. *What does this mean?*

Remember by the definition of the last section:

Definition (I.5). The **set** $A \setminus B$ is $A \setminus B = \{x \mid x \in A,\ x \notin B\}$

Thus $x \in (A \setminus B)$ means that $x \in A$ but $x \notin B$. Since $x \in A$ therefore $(A \setminus B) \subseteq A$.

■

B4 Equal Sets

This subsection is also challenging because you are required to prove certain results which means that you will need to know your definitions thoroughly.

We can use the concept of subsets to show 2 sets are equal.

Definition (I.11). Two sets A and B are equal if and only if $A \subseteq B$ and $B \subseteq A$.

We can use this definition to prove two sets, A and B, are equal, $A = B$, by showing $A \subseteq B$ and $B \subseteq A$. This means that sets A and B are equal if and only if A is a subset of B and B is a subset of A.

This is similar to proving 2 numbers are equal. To show that numbers a and b are equal we show $a \le b$ and $b \le a$ then $a = b$.

The following is an example of an application of the definition.

Example 15

Given the set A. Prove that $A = A \cup A$.

How do we prove this result?

Let x be an arbitrary member of the set A and show that it is also in $A \cup A$. Then assume an arbitrary member y is an element of the set $A \cup A$ and show that y is also a member of the set A. This approach shows $A \subseteq (A \cup A)$ and $(A \cup A) \subseteq A$ which by the above definition (I.11) proves $A = A \cup A$.

Proof.

Let x be an arbitrary member of the set A. Therefore x is a member of $A \cup A$ which means that $A \subseteq (A \cup A)$.

Now let y be an arbitrary member of the set $A \cup A$ which means it is a member of the set A therefore we conclude that $(A \cup A) \subseteq A$.

Combining these two, $A \subseteq (A \cup A)$ and $(A \cup A) \subseteq A$, and using definition (I.11) we have our result $A = A \cup A$.

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Example 16

Given the set A. Prove that $(A^c)^c = A$.

How do we prove this result?

To show equality of sets we need to prove that $(A^c)^c \subseteq A$ and $A \subseteq (A^c)^c$.

Proof.

Let x be an arbitrary element of $(A^c)^c$ that is $x \in (A^c)^c$. *What does this mean?*

By Definition (I.4) of the last section $x \in (A^c)^c$ means $x \notin A^c$. This means $x \in A$ because x is **not** a member of A^c:

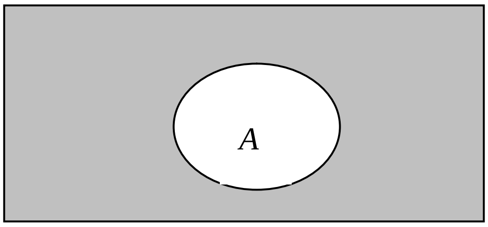

Figure 18 A^c shaded

Thus we have $(A^c)^c \subseteq A$. *How do we continue with the proof?*

We go the other way, that is we show $A \subseteq (A^c)^c$.

Let y be an arbitrary member of the set A. This means that y is **not** a member of A^c, that is $y \notin A^c$ which means that $y \in (A^c)^c$. Thus we have $A \subseteq (A^c)^c$.

Combining the two results together, $(A^c)^c \subseteq A$ and $A \subseteq (A^c)^c$, we have $(A^c)^c = A$.

This completes the proof.

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SUMMARY

The set A is a subset of B if every element of A is also in the set B. This is denoted by $A \subseteq B$.

The **power set** of A is the set of **all** subsets of the set A and is denoted by $P(A)$.

The cardinality of a set A is the number of elements in the set and is denoted by $|A|$.

To prove that $A \subseteq B$ we consider an arbitrary element in A and show that it is also in B.

To prove that two sets are equal, $A = B$, we show that $A \subseteq B$ and $B \subseteq A$.