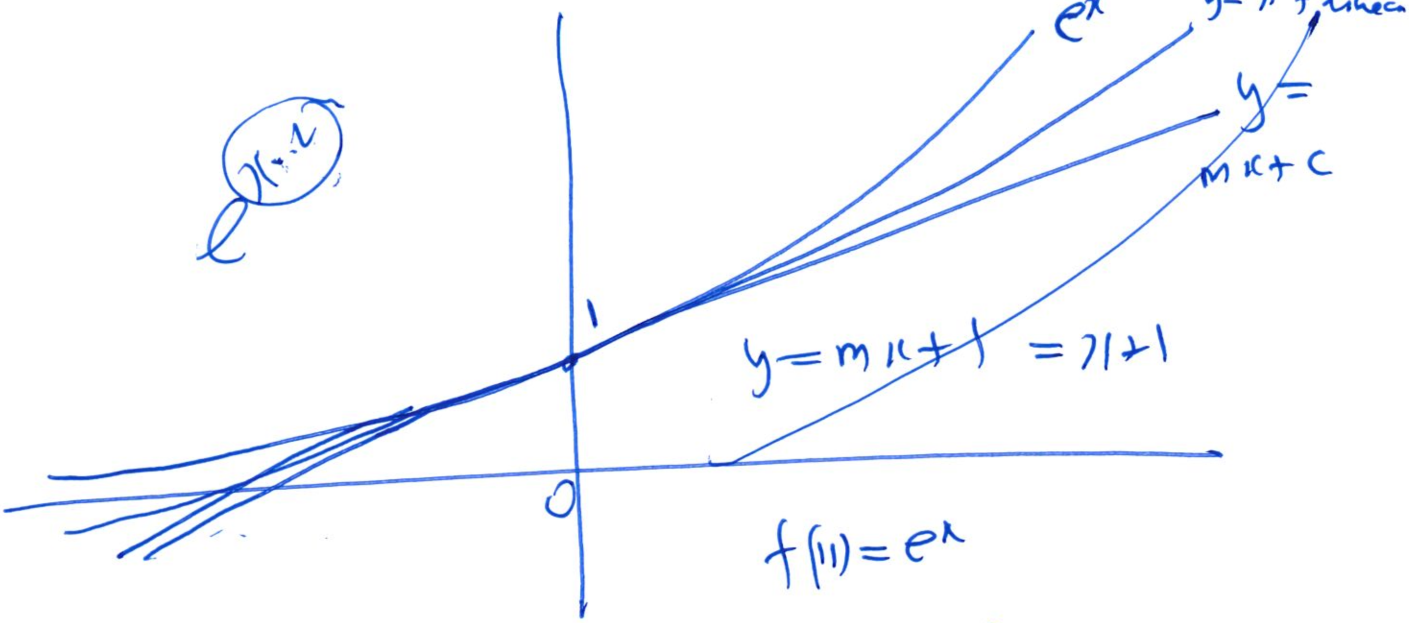
Series Expansion Page 358 (Chapter 7).



$f(x) = e^x$

$f'(x) = e^x$

subs $x = 0$ $\quad f'(0) = e^0 = 1$

(*) $\underline{f(x) = A + Bx + Cx^2 + Dx^3 + Ex^4 + \dots}$

Substituting $x = 0$ gives

$$f(0) = A$$

Differentiating (*) gives

$$\underline{f'(x) = 0 + B + 2Cx + 3Dx^2 + 4Ex^3 + \dots}$$

Substituting $x = 0$ gives

$$\cancel{f'(0) = 2B \Rightarrow C = \frac{f'(0)}{2}}$$

$$f'(0) = B$$

$(**)$ $f''(x) = 2C + 3(2)Dx + 4(3)Ex^2 + \cdots$

$$f''(0) = 2C + 3(2)D(0) + 4 \cdot 3E(0^2) + \cdots$$

$$C = \frac{f''(0)}{2!}$$

Differentiating $(**)$ gives

$$f'''(x) = 3(2)D + 4(3)(2)Ex + \cdots$$

$x = 0$: $\quad f'''(0) = 3(2)D + \underbrace{4(3)(2)E(0)}_{=0}$

$$D = \frac{f'''(0)}{3 \times 2} = \frac{f'''(0)}{3!}$$

Diff again:

$$f^{iv}(x) = 0 + 4(3)(2)E + \cdots$$

$$E = \frac{f^{iv}(0)}{4(3)(2)} = \frac{f^{iv}(0)}{4!}$$

$A = f(0)$, $B = f'(0)$, --

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{iv}(0)}{4!}x^4 +$$

Find the Mac series for e^x.

Sol^n: Let $f(x) = e^x$. $\quad f(0) = e^0 = 1$

$f'(x) = e^x \quad f'(0) = 1$

$f''(x) = e^x \quad f''(0) = 1$

$f'''(x) = e^x \quad f'''(0) = 1$

$f^{iv}(x) = e^x \quad f^{iv}(0) = 1$

$f^{v}(x) = e^x \quad f^{v}(0) = 1$

$$f(x) = f(0) + x f'(0) + \frac{f''(0)}{2!} x^2 +$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

Find the series expansion of $\sin(x)$.

Sol^n: Let $f(x) = \sin(x) \quad f(0) = \sin(0) = 0$

$f'(x) = \cos(x) \quad f'(0) = \cos(0) = 1$

$f''(x) = -\sin(x) \quad f''(0) = 0$

$f'''(x) = -\cos(x) \quad f'''(0) = -1$

$f^{iv}(x) = \sin(x) \quad f^{iv}(0) = 0$

$f^{v}(x) = \cos(x) \quad f^{v}(0) = 1$

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots$$

$$\sin(x) = 0 + x + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-1) + \frac{x^4}{4!}(0) + \frac{x^5}{5}(1)$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\cos(1) = 1 - \frac{1^2}{2!} + \frac{1^4}{4!} - \frac{1^6}{6!} + \frac{1^8}{8!} - \cdots$$

$$= 0.540303$$

$$= 0.540302 \quad (\text{by cal})$$

$$\lim_{x \to 0} \frac{\sin(x)}{x} = \lim_{x \to 0} \left(\frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots}{x} \right)$$

$$= \lim_{x \to 0} \frac{\cancel{x}\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\right)}{\cancel{x}}$$

$$= \lim_{x \to 0} \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!}\right) = 1$$