SECTION D **Principle of Mathematical Induction**

By the end of this section you will be able to

- understand the procedure for proof by induction
- construct proofs by induction

In this section we examine propositions concerning positive integers. The positive integers 1, 2, 3, 4, ,… are called **natural numbers** or **counting numbers**. In this section, lower case letters represent natural numbers.

D1 Principle of Mathematical Induction

Mathematical induction is a powerful tool used to prove propositions concerning natural numbers.

Principle of Mathematical Induction (I.15)

For each natural number n, let $P(n)$ be a proposition about n. If $P(n)$ satisfies:

1) $P(1)$ is true,
2) for an arbitrary k, $P(k)$ is true implies $P(k+1)$ is true.

Then for *all* natural numbers, n , we have $P(n)$ is true.

Parts 1) and 2) suggest that

$$P(1) \text{ implies } P(2),\ P(2) \text{ implies } P(3),\ \cdots,\ P(k) \text{ implies } P(k+1),...$$

This is called the domino effect. Once one of the dominos topples it causes the rest to topple as well. You use the k th domino to knock down the $(k+1)$th domino.

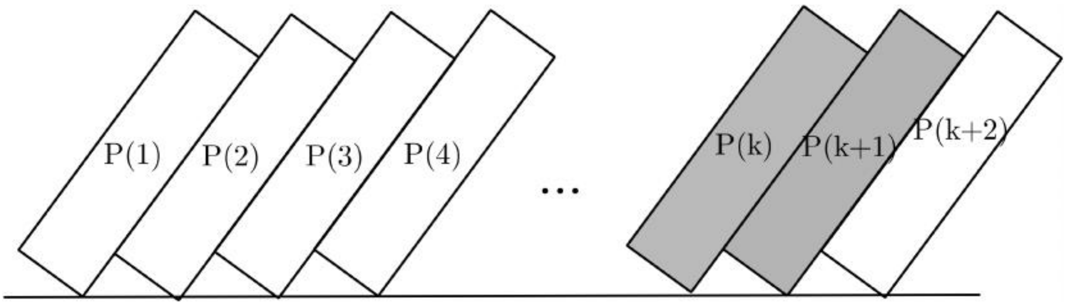

Figure 7 Domino Effect

Another analogy is climbing an infinite rung ladder. Mathematical induction says if you can climb onto the first step of the ladder, and from any (k th) step you can climb to the next $(k+1)$th step, then you can climb to the top of the ladder, although this ladder has an infinite number of rungs.

The process is that we show $P(1)$ is true and by assuming $P(k)$ is true we prove $P(k+1)$ is true. If *both* $P(1)$ is true and $P(k)$ implies $P(k+1)$ then proposition $P(n)$ is true for all natural numbers n.

We can apply this principle of mathematical induction to prove results about natural numbers.

Example 29

For every natural number n prove the proposition $P(n)$ given by

$$1+2+3+4+...+n=\frac{1}{2}n(n+1).$$

What does this proposition mean?

It means that if we add the first n natural numbers then the answer will be $\frac{1}{2}n(n+1)$. For example, if we add the first two numbers we have

$$1+2=\frac{1}{2}(2)(2+1)=3 \qquad \text{[Substituting } n=2 \text{ into the above]}$$

We need to show this result for all the natural numbers n. *How?*

We use mathematical induction because the proposition concerns the natural numbers.

Proof.

First, we check the proposition for $n=1$:

$$1=\frac{1}{2}(1)(1+1) \qquad \checkmark$$

Hence the proposition is true for $n=1$. Next, we *assume* the given proposition is true for $n=k$, that is $P(k)$. *How do we write this* $P(k)$*?*

By substituting $n=k$ into the given proposition

$$1+2+3+4+...+n=\frac{1}{2}n(n+1)$$

which yields

$$1+2+3+4+...+k=\frac{1}{2}(k)(k+1) \qquad (*)$$

We have labelled this result by (*) because we are going to prove the proposition for $n=k+1$ by using (*). *How do we write the proposition* $P(k+1)$*?*

This is the addition of the first $k+1$ natural numbers, so substituting $n=k+1$ into the given proposition gives:

$$1+2+3+4+...+k+(k+1)=\frac{1}{2}(k+1)\underbrace{(k+1+1)}_{=k+2}=\frac{1}{2}(k+1)(k+2) \qquad (**)$$

This means that *we have to prove* the sum of the first $(k+1)$ natural numbers is equal to $\frac{1}{2}(k+1)(k+2)$. It is critical that you realise we need to prove (**), we have *only* stated $P(k+1)$ *not* proven it yet. The challenge is to show that the left-hand side is equal to the right-hand side of (**). *How?*

By simplifying the left – hand side using (*);

$$\begin{aligned}
1+2+3+\ldots+k+(k+1) &= \underbrace{1+2+3+4+\ldots+k}_{=\frac{1}{2}k(k+1) \text{ by } (*)}+(k+1) \\
&= \frac{1}{2}k(k+1)+(k+1) && [\text{Simplifying}] \\
&= \frac{1}{2}(k+1)k+\frac{1}{2}(k+1)2 && \left[\text{Rewriting } (k+1)=\frac{1}{2}(k+1)2\right] \\
&= \frac{1}{2}(k+1)(k+2) && [\text{Factorizing}]
\end{aligned}$$

The last line is the right-hand side of (**). Hence our result holds by the Principle of Mathematical Induction (I.15) because we have shown (**).

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Notice how we assume $P(k)$ to be true and then use it to prove $P(k+1)$. The proposition $P(k)$ in the above was (*) and we used this in the derivation of $P(k+1)$. We knock down the first domino $P(1)$ and then we use the k th domino $P(k)$ to knock down the $(k+1)$th domino $P(k+1)$. By mathematical induction we conclude that adding the first n positive integers gives us:

$$1+2+3+4+\ldots+n=\frac{1}{2}n(n+1).$$

We can write the left – hand side $1+2+3+4+\ldots+n$ in compact mathematical notation. In mathematics we use the Greek letter $\sum$, pronounced sigma, for writing this:

$$1+2+3+4+\ldots+n=\sum_{m=1}^{n} m.$$

This $\sum_{m=1}^{n} m$ means 'the sum of all positive whole numbers between 1 and n'.

We can write the sum of odd numbers as $1+3+5+\ldots+(2n-1)=\sum_{m=1}^{n}(2m-1)$.

Example 30

For every natural number n prove the proposition $P(n)$ given by

$$1+3+5+7+\ldots+(2n-1)=\sum_{m=1}^{n}(2m-1)=n^2$$

The proposition says that if we add the first n odd counting numbers then the answer will be the square of n. For example, if we add the first two odd counting numbers we have

$$1+3=2^2=4 \qquad \text{[Substituting } n=2 \text{ into the given proposition]}$$

Similarly, adding the first five odd counting numbers gives

$$\underbrace{1+3+5+7+9}_{\text{5 Terms}}=5^2 \qquad \left[\text{Adding the first 5 odd numbers, that is } n=5\right]$$

and so on.

Proof.

First, we check the proposition for $n=1$:

$$1=1^2 \quad \checkmark$$

The proposition is true for $P(1)$. Next, we *assume* the given proposition is true for $n=k$ that is $P(k)$:

$$\underbrace{1+3+5+7+\ldots+(2k-1)}_{\text{first } k \text{ odd counting numbers}}=k^2 \qquad (\dagger)$$

We have labelled $P(k)$ by (†) because we are going to prove the proposition for $n=k+1$ by using (†). *How do we write the proposition* $P(k+1)$*?*

By adding the first $k+1$ odd counting numbers (substituting $n=k+1$):

$$\underbrace{1+3+5+7+\ldots+(2k-1)}_{\text{First } k \text{ terms}}+\underbrace{\left(2(k+1)-1\right)}_{(k+1)\text{th term}}=(k+1)^2 \qquad (\dagger\dagger)$$

We need to prove this, (††), result. *How?*

We can simplify the sum from 1 to $(2k-1)$ by using (†), we have

$$\begin{aligned}\underbrace{1+3+5+\ldots+(2k-1)}_{\text{First } k \text{ odd terms}}+\left(2(k+1)-1\right) &= \underbrace{1+3+5+7+\ldots+(2k-1)}_{=k^2 \text{ by } (\dagger)}+(2k+2-1)\\ &= k^2+2k+1 && \left[\text{Simplifying}\right]\\ &= (k+1)^2 && \left[\text{Factorizing}\right]\end{aligned}$$

The last line is identical to the right-hand side of (††). By the Principle of Mathematical Induction, we have the sum of first n odd counting numbers is n^2.

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In the above example, we first showed that the given result was true for $P(1)$:

$$1=1^2 \qquad \left[n=1\right]$$

Secondly, we assumed it is true for $P(k)$:

$$1+3+5+7+\ldots+(2k-1)=k^2 \qquad \left[n=k\right]$$

and finally, we used this assumption to produce the result for $P(k+1)$:

$$1+3+5+7+...+(2k-1)+(2(k+1)-1)=(k+1)^2 \qquad [n=k+1]$$

G3 Divisibility Example

We can apply mathematical induction to divisibility examples.

Remember we can write a divides b by the following notation $a \mid b$ where the vertical line represents division. Recall a divides b means that there is an integer m such that $am=b$ or b is a multiple of a. In mathematical notation we write

$$a \mid b \quad \Leftrightarrow \text{ there is an integer } m \text{ such that } am=b.$$

This was Definition (I.5) earlier in the chapter.

Example 31

For every natural number n prove the proposition $P(n)$ given by

$$3 \mid (2^{2n-1}+1).$$

What does $3 \mid (2^{2n-1}+1)$ *mean?*

$$2^{2n-1}+1 \text{ is divisible by 3 exactly}$$

or there is an integer m such that

$$2^{2n-1}+1=3m$$

To put it another way, $2^{2n-1}+1$ is a multiple of 3 for every natural number n.

Proof.

How do we prove $3 \mid (2^{2n-1}+1)$ *?*

We apply mathematical induction. *Why?*

Because the given proposition $P(n)$ holds for every natural number n.

First we check the first domino $P(1)$ is knocked over, that is substituting $n=1$ into $2^{2n-1}+1$:

$$2^{2-1}+1=2^1+1=3 \qquad \checkmark$$

Clearly 3 divides 3 and this is denoted by $3 \mid (2^{2-1}+1)$. Hence the proposition is true for $P(1)$. Next we assume the given proposition is true for $n=k$ that is 3 divides $2^{2k-1}+1$ or in notation form $3 \mid (2^{2k-1}+1)$. This means there is an integer q such that

$$3q=2^{2k-1}+1 \qquad (\$)$$

The challenge is to prove the result for $n=k+1$ by using (\$). *How do we write down* $P(k+1)$ *?*

By substituting $n = k+1$ into the given proposition $3 \mid \left(2^{2n-1}+1\right)$:

$$3 \mid \left(2^{2(k+1)-1}+1\right).$$

That is, we need to prove

$$3 \text{ divides } 2^{2(k+1)-1}+1.$$

Let's examine the right - hand term, $2^{2(k+1)-1}+1$:

$$\begin{aligned} 2^{2(k+1)-1}+1 &= 2^{2k-1+2}+1 && \left[\text{Rewriting the index of 2}\right] \\ &= 2^{2k-1}2^{2}+1 && \left[\text{Applying the rules of indices } a^{m+n}=a^{m}a^{n}\right] \\ &= (4)2^{2k-1}+1 && \left[\text{Rewriting } 2^{2}=4\right] \\ &= (3+1)2^{2k-1}+1 && \left[\text{Rewriting } 4=3+1\right] \\ &= (3)2^{2k-1}+2^{2k-1}+1 && \left[\text{Expanding } (3+1)2^{2k-1}\right] \end{aligned}$$

By ($) we know the last two terms on the right-hand side, $2^{2k-1}+1$, are equal to $3q$. Therefore, we obtain

$$\begin{aligned} 2^{2(k+1)-1}+1 &= (3)2^{2k-1}+\underbrace{2^{2k-1}+1}_{=3q \text{ by } (\$)} \\ &= (3)2^{2k-1}+3q \\ &= 3\left(2^{2k-1}+q\right) && \left[\text{Taking out a common factor of 3}\right] \end{aligned}$$

Thus the left - hand term $2^{2(k+1)-1}+1 = 3(\text{Integer})$ which means it is a multiple of 3 or 3 divides $2^{2(k+1)-1}+1$. We have proven $P(k) \Rightarrow P(k+1)$, therefore our result follows by induction.

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G4 Factorization Example

We can apply the principle of mathematical induction to prove general results concerning natural numbers. For example, we can use induction to prove the binomial theorem for positive integers (natural numbers). There is a great deal of algebraic manipulation in proving the binomial theorem, but the procedure of mathematical induction is the same. You are asked to show the binomial theorem in question 18 of Exercise I(d).

Let's first prove a result regarding factorizing of $a^{n}-b^{n}$ where a and b are real numbers and n is a natural number. This is a particularly useful result because it can be employed to factorize expressions which look like $a^{n}-b^{n}$. The difficulty is trying to prove the result for $n=k+1$ and we use the 'trick' of writing 0 as $x-x$ or in our Example 32 below as $-a^{k}b+a^{k}b(=0)$.

Up to now we have been proving results by mathematical induction for all natural numbers 1, 2, 3, 4, …, n, …

Clearly some results may *not* be valid for the first few natural numbers. That is the starting point may *not* be 1 but some other natural number such as n_0 say. In the next example the result is valid for 2, 3, 4, 5,…, n, … so the starting point is $n = 2$ and *not* $n = 1$. In general, the process of mathematical induction is the same apart from the starting point. If the starting point is n_0 then the process of mathematical induction is:

1. We check the result for $n = n_0$ (starting point). Check $P(n_0)$.
2. Assume it is true for $n = k$. Assume $P(k)$.
3. Prove $P(k) \Rightarrow P(k+1)$.

We use this to show the following identity:

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + b^{n-1})$$

What does the term identity mean?

An **identity** is an algebraic expression which is valid for all values of the unknowns. For example, the following are identities:

$(a+b)^2 = a^2 + 2ab + b^2$ is true for all values of a and b.

$a^2 - b^2 = (a-b)(a+b)$ is true for all values of a and b.

$\sin^2(\theta) + \cos^2(\theta) = 1$ is true for all values of θ.

$2x + 1 = 0$ is *not* an identity because it is only true when $x = -1/2$.

Example 32

Let a and b be real numbers then for the natural numbers $n \geq 2$ we have the proposition $P(n)$ given by

(I.16) $$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots + ab^{n-2} + b^{n-1})$$

Prove $P(n)$.

What does this proposition mean?

An expression of the form $a^n - b^n$ can be factorized into

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots + ab^{n-2} + b^{n-1})$$

and of course, we can use this to solve equations of the type $a^n - b^n = 0$. *In any case how do we prove this result?*

Since it is a result concerning natural numbers n therefore, we can use induction.

Proof.

We first show this result for $n=2$ (Our starting point is $n=2$). *How?*

By substituting $n=2$ into the given proposition:

$$a^2-b^2=(a-b)(a+b)$$

Of course, this is a fundamental identity of algebra, do you remember what it is called?

Difference of two squares. Thus $P(2)$ is true.

Assume the proposition is true for $n=k$ that is $P(k)$:

$$a^k-b^k=(a-b)\left(a^{k-1}+a^{k-2}b+a^{k-3}b^2+\dots+ab^{k-2}+b^{k-1}\right) \qquad (*)$$

The difficulty in the process of induction is to prove the result for $n=k+1$ by employing (*). *What do we need to prove?*

Required to prove $P(k+1)$, that is:

$$\begin{aligned} a^{k+1}-b^{k+1}&=(a-b)\left(a^{k+1-1}+a^{k+1-2}b+a^{k+1-3}b^2+\dots+ab^{k+1-2}+b^{k+1-1}\right)\\ &=(a-b)\left(a^k+a^{k-1}b+a^{k-2}b^2+\dots+ab^{k-1}+b^k\right) \qquad (**)\end{aligned}$$

We need to show the left-hand side is equal to the right-hand side of (**). Let's consider the left-hand side on its own:

$$\begin{aligned} a^{k+1}-b^{k+1}&=a^{k+1}-a^kb+a^kb-b^{k+1} && \left[\begin{array}{l}\text{Using the above stated trick}\\ \text{of writing } 0=-a^kb+a^kb\end{array}\right]\\ &=a^ka-a^kb+a^kb-b^kb && \left[\text{Using the rules of indices } a^{m+n}=a^ma^n\right]\\ &=a^k(a-b)+b\left(a^k-b^k\right) && \left[\text{Factorizing out common terms}\right]\\ &=a^k(a-b)+b\underbrace{(a-b)\left(a^{k-1}+a^{k-2}b+a^{k-3}b^2+\dots+ab^{k-2}+b^{k-1}\right)}_{\text{by }(*)}\\ &=(a-b)\left(a^k+b\left(a^{k-1}+a^{k-2}b+a^{k-3}b^2+\dots+ab^{k-2}+b^{k-1}\right)\right) && \left[\text{Factorizing out } (a-b)\right]\\ &=(a-b)\left(a^k+a^{k-1}b+a^{k-2}b^2+a^{k-3}b^3+\dots+ab^{k-1}+b^k\right) && \left[\begin{array}{l}\text{Multiplying by } b\\ \text{in the second bracket}\end{array}\right]\end{aligned}$$

Since the last line is the right-hand side of (**) we have shown (**). Hence, we have our factorized result, $a^n-b^n=(a-b)\left(a^{n-1}+a^{n-2}b+a^{n-3}b^2+\dots+ab^{n-2}+b^{n-1}\right)$.

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Example 32 was a challenging problem but the procedure for mathematical induction is the same apart from the starting point which was $n=2$.

In the next example we prove a proposition which is a **corollary**.

A corollary is a proposition which follows from one already proved. Normally it is a special case of the already proven proposition.

Example 33

Prove the following:

Corollary (I.17) $a^{rs}-1=\left(a^{r}-1\right)\left(a^{r(s-1)}+a^{r(s-2)}+\cdots+a^{r}+1\right)$

where r and s are integers.

Note that this corollary says that $a^{rs}-1$ factorizes into

$$\left(a^{r}-1\right)\left(a^{r(s-1)}+a^{r(s-2)}+\cdots+a^{r}+1\right)$$

Proof.

Using the rules of indices to rewrite $a^{rs}=\left(a^{r}\right)^{s}$ and the previous proposition:

(I.16) $a^{n}-b^{n}=\left(a-b\right)\left(a^{n-1}+a^{n-2}b+a^{n-3}b^{2}+\ldots+ab^{n-2}+b^{n-1}\right)$

with $a=a^{r}$, $b=1$ and $n=s$ gives

$$\begin{aligned}\left(a^{r}\right)^{s}-1^{s}&=\left(a^{r}-1\right)\left(\left(a^{r}\right)^{s-1}+\left(a^{r}\right)^{s-2}1+\left(a^{r}\right)^{s-3}1^{2}+\ldots a^{r}1^{s-2}+1^{s-1}\right)\\&=\left(a^{r}-1\right)\left[a^{r(s-1)}+a^{r(s-2)}+a^{r(s-3)}+\cdots+a^{r}+1\right]\end{aligned}$$

Therefore $a^{rs}-1=\left(a^{r}-1\right)\left(a^{r(s-1)}+a^{r(s-2)}+\cdots+a^{r}+1\right)$. This completes our proof.

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Note that this corollary is the special case with a being replaced by a^{r}, b being replaced by 1 and n being replaced by s in the previous general proposition (I.16).

SUMMARY

We use mathematical induction to prove propositions involving natural numbers.

The principle of mathematical induction to prove a proposition $P(n)$ involves:

1. Showing the result for $n=n_{0}$, that is $P(n_{0})$. This is the base case.
2. Assuming the result is true for $n=k$ where k is an arbitrary positive integer, that is assuming $P(k)$ is true.
3. Prove the result for $n=k+1$, that is prove $P(k+1)$.