Exercise **1(h)**

1. Prove that for all natural numbers n, 9 divides $10^n - 1$.

2. Prove that for all natural numbers n,
$$3 \mid \left(n^3 - n\right)$$

3. By using mathematical induction show that for every natural number n:
$$3 \mid n(n+1)(n+2)$$

4. Prove that for all natural numbers n:
$$n^2 - n \text{ is an even number}$$

5. Prove that for all natural numbers n
$$a + ar + ar^2 + \ldots + ar^{n-1} = \frac{a\left(1-r^n\right)}{1-r} \qquad [r \neq 1]$$
where a and r are real numbers.
[This is a geometric series with first term equal to a and common ratio r]

6. Let $a \mid b$. Prove by mathematical induction that $a^n \mid b^n$ for all $n \in$.

7. Find the fallacy in the following proof of:
For all $n \in$
$$n \geq 2^n - 1$$
"Proof"
The result is true for $n = 1$. Assume it is true for $n = k$:
$$k \geq 2^k - 1 \qquad (*)$$
Also for $n = k + 1$:
$$k + 1 \geq 2^{k+1} - 1 \qquad (**)$$
Subtracting (*) from (**) gives
$$1 \geq 2^{k+1} - 1 - \left(2^k - 1\right) = 2^k$$

8. *(i) Prove that for all $n \geq 3$:
$$3n^2 + 3n + 1 < 2n^3$$
(ii) Prove that for all $n \geq 4$:
$$3^n > n^3$$

9. (i) Prove that $n^2 \geq 2n + 1$ for $n \geq 3$.
(ii) Prove $2^n \geq n^2$ for all natural numbers $n \geq 4$.

10. *Prove Bernoulli's inequality; For real number $x > -1$ and natural number n we have

$$(1+x)^n \geq 1+nx$$

11. *Prove that for all natural numbers n we have the following trigonometric identity:

$$\sin(x)+\sin(2x)+...+\sin(nx)=\frac{\cos\left(\frac{x}{2}\right)-\cos\left(\frac{2n+1}{2}x\right)}{2\sin\left(\frac{x}{2}\right)}$$

where x is a real number such that $\sin\left(\frac{x}{2}\right)\neq 0$.

12. **Prove the binomial theorem for the natural number n:
If a and b are real numbers then the binomial theorem says that for all natural numbers n

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \cdots + b^n$$

For the following proof you will need to use the strong induction process.

13. **Prove there are an infinite number of primes using mathematical induction.
[Hint: Assume there are $p_1,\ p_2,\ \cdots,\ p_k$ primes and then consider the number $(p_1 \times p_2 \times \cdots \times p_k)+1$.]