Complete Solutions to Sample Exam and Test Questions

1. The given function is odd so we only have sine terms in the Fourier series of $f(t)$. We have $A_0 = 0,\ A_k = 0$ and

$$\begin{aligned}B_k &= \frac{2}{\pi}\int_0^{\pi} 2\sin(kt)\,\mathrm{d}t \\ &= \frac{4}{\pi}\int_0^{\pi}\sin(kt)\,\mathrm{d}t \\ &= -\frac{4}{\pi}\left[\frac{\cos(kt)}{k}\right]_0^{\pi} \\ &= -\frac{4}{k\pi}\left[\cos(k\pi)-\cos(0)\right] \\ &= -\frac{4}{k\pi}\left[\cos(k\pi)-1\right]\end{aligned}$$

Recall that

$$\cos(k\pi) = \begin{cases} 1 & \text{if } k = \text{even} \\ -1 & \text{if } k = \text{odd}\end{cases}$$

Putting this into the above calculation $B_k = -\frac{4}{k\pi}\left[\cos(k\pi)-1\right]$ gives

$$B_k = -\frac{4}{k\pi}\left[\cos(k\pi)-1\right] = -\frac{4}{k\pi}\begin{cases}\left[1-1\right]=0 & \text{if } k=\text{even} \\ \left[-1-1\right]=-2 & \text{if } k=\text{odd}\end{cases}$$

Hence $B_k = 0$ if k is even and $B_k = -\frac{4}{k\pi}\left[-2\right] = \frac{8}{k\pi}$ if k is odd.

Substituting this into the general Fourier series

(17.2) $\quad f(t) = A_0 + A_1\cos(t) + A_2\cos(2t) + \ldots + B_1\sin(t) + B_2\sin(2t) + \ldots$

Gives

$$\begin{aligned}f(t) &= 0 + 0 + \ldots + \frac{8}{\pi}\sin(t) + 0 + \frac{8}{3\pi}\sin(3t) + 0 + \frac{8}{5\pi}\sin(5t) + \cdots \\ &= \frac{8}{\pi}\left[\sin(t) + \frac{\sin(3t)}{3} + \frac{\sin(5t)}{5} + \cdots\right]\end{aligned}$$

2. (i) The given series is odd so it only has sine terms which are given by

$$\begin{aligned} B_k &= \frac{2}{\pi}\int_0^{\pi}\left[f\left(t\right)\sin\left(kt\right)\right]\mathrm{d}t \\ &= \frac{2}{\pi}\int_0^{\pi}\left[2t\sin\left(kt\right)\right]\mathrm{d}t \qquad \left[\text{Substituting } f\left(t\right)=2t\right] \\ &= \frac{4}{\pi}\int_0^{\pi}\left[t\sin\left(kt\right)\right]\mathrm{d}t \qquad (*) \end{aligned}$$

How do we integrate $\int_0^{\pi}\left[t\sin\left(kt\right)\right]\mathrm{d}t$ *?*

Use integration by parts with:

$$u = t \qquad v' = \sin\left(kt\right)$$

$$u' = 1 \quad [\text{Differentiating}] \qquad v = \int \sin\left(kt\right)\mathrm{d}t = -\frac{\cos\left(kt\right)}{k}$$

Substituting this into the integration by parts formula gives:

$$\begin{aligned} \int_0^{\pi}\left[t\sin\left(kt\right)\right]\mathrm{d}t &= uv - \int u'v\,\mathrm{d}t \\ &= -\left[\frac{t\cos\left(kt\right)}{k}\right]_0^{\pi} - \int_0^{\pi}\left(1\right)\left(-\frac{\cos\left(kt\right)}{k}\right)\mathrm{d}t \\ &= -\left[\frac{\pi\cos\left(k\pi\right)}{k} - 0\right] + \left[\frac{\sin\left(kt\right)}{k^2}\right]_0^{\pi} \\ &= -\frac{\pi\cos\left(k\pi\right)}{k} + \underbrace{\left[\frac{\sin\left(k\pi\right)-\sin\left(0\right)}{k^2}\right]}_{=0} \qquad \left[\begin{matrix}\text{Because} \\ \sin\left(k\pi\right)=\sin\left(0\right)=0\end{matrix}\right] \end{aligned}$$

$$\int_0^{\pi}\left[t\sin\left(kt\right)\right]\mathrm{d}t = -\frac{\pi\cos\left(k\pi\right)}{k}$$

Substituting this $\int_0^{\pi}\left[t\sin\left(kt\right)\right]\mathrm{d}t = -\frac{\pi\cos\left(k\pi\right)}{k}$ into (*) gives

$$B_k = \frac{4}{\cancel{\pi}}\left[-\frac{\cancel{\pi}\cos\left(k\pi\right)}{k}\right] = -\frac{4}{k}\cos\left(k\pi\right)$$

If k is odd then $\cos\left(k\pi\right) = -1$ and we have

$$B_k = -\frac{4}{k}\left(-1\right) = \frac{4}{k} \quad (\text{odd } k)$$

If k is even then $\cos\left(k\pi\right) = 1$ and we have

$$B_k = -\frac{4}{k}\left(1\right) = -\frac{4}{k} \quad (\text{even } k)$$

Hence the Fourier series is given by

$$f(t) = \underbrace{0}_{\text{constant}} + \underbrace{0}_{\substack{\text{No cosine}\\\text{terms}}} + 4\sin(t) - \frac{4\sin(2t)}{2} + \frac{4\sin(3t)}{3} - \frac{4\sin(4t)}{4} + \ldots$$

$$= 4\left[\sin(t) - \frac{\sin(2t)}{2} + \frac{\sin(3t)}{3} - \frac{\sin(4t)}{4} + \ldots\right] \qquad \left[\text{Taking out } 4\right]$$

(ii) We need to show that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$.

Substituting $t = \frac{\pi}{2}$ into the above derived Fourier series gives

$$f\left(\frac{\pi}{2}\right) = 4\left[\sin\left(\frac{\pi}{2}\right) - \frac{\sin(2\pi/2)}{2} + \frac{\sin(3\pi/2)}{3} - \frac{\sin(4\pi/2)}{4} + \frac{\sin(5\pi/2)}{5} - \ldots\right]$$

$$\pi = 4\left[1 - \frac{0}{2} + \frac{-1}{3} - \frac{0}{4} + \frac{1}{5} - \ldots\right]$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

We have shown our required result.

3. We have $F(x) = \sum_{k=1}^{\infty} B_k \sin(kx)$

where $$B_k = \frac{2}{\pi}\int_0^{\pi} 1\sin(kx)\,\mathrm{d}x = -\frac{2}{\pi}\left[\frac{\cos(kx)}{k}\right]_0^{\pi} = \frac{2}{k\pi}\left(1 - \cos(k\pi)\right)$$

Using

$$\cos(k\pi) = \begin{cases} 1 & \text{if } k = \text{even} \\ -1 & \text{if } k = \text{odd} \end{cases}$$

Gives

$$B_k = \frac{2}{k\pi}\left(1 - \cos(k\pi)\right) = \begin{cases} \frac{2}{k\pi}(1-1) = 0 & \text{if } k = \text{even} \\ \frac{2}{k\pi}\left(1-(-1)\right) = \frac{4}{k\pi} & \text{if } k = \text{odd} \end{cases}$$

Therefore $B_k = 0$ if k is even, and $B_k = \frac{4}{k\pi}$ if k is odd

We thus have, in $0 < x < ®$,

$$1 = \frac{4}{\pi}\left(\sin\left(x\right) + \frac{\sin\left(3x\right)}{3} + \frac{\sin\left(5x\right)}{5} + \frac{\sin\left(7x\right)}{7} + \ldots\right)$$

We can write this in compact form as

$$1 = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin\left(\left(2n-1\right)x\right)$$

4. The function can be defined as $f(x) = \frac{5x}{\circledR}$, $-\circledR < x < \circledR$. This is the long answer if you did *not* spot that the given function is odd.

$$a_0 = \frac{1}{\circledR}\int_{-\circledR}^{\circledR} f(x)dx = \frac{1}{\circledR}\int_{-\circledR}^{\circledR}\frac{5x}{\circledR}dx = 0 \qquad \underline{\therefore a_o = 0}$$

$$a_n = \frac{1}{\circledR}\int_{-\circledR}^{\circledR} f(x)\cos nx\,dx = \frac{1}{\circledR}\int_{-\circledR}^{\circledR}\frac{5x}{\circledR}\cos nx\,dx = \frac{5}{\circledR^2}\int_{-\circledR}^{\circledR} x\cos nx\,dx$$

$$a_n = \frac{5}{\circledR^2}\left\{\left[x\frac{\sin nx}{n}\right]_{-\circledR}^{\circledR} - \frac{1}{n}\int_{-\circledR}^{\circledR}\sin nx dx\right\}$$

$$a_n = \frac{5}{\circledR^2}\left\{\left[\frac{x\sin nx}{n}\right]_{-\circledR}^{\circledR} + \frac{1}{n}\times 0\right\} = 0 \qquad \underline{\therefore a_n = 0}$$

$$b_n = \frac{1}{\circledR}\int_{-\circledR}^{\circledR} f(x)\sin nx\,dx = \frac{1}{\circledR}\int_{-\circledR}^{\circledR}\frac{5x}{\circledR}\sin nx\,dx\,,\; b_n = \frac{5}{\circledR^2}\left\{\left[x\left(\frac{-\cos nx}{n}\right)\right]_{-\circledR}^{\circledR} + \frac{1}{n}\int_{-\circledR}^{\circledR}\cos nx\right\}$$

$$b_n = \frac{5}{\circledR^2}\left\{\frac{-\circledR\cos n\circledR}{n} - \frac{\circledR\cos n\circledR}{n} + 0\right\} = -\frac{10}{\circledR}\frac{\cos n\circledR}{n}$$

For n even $\cos n\circledR = 1$ $\qquad \underline{\therefore b_n = -\frac{10}{n\circledR}}$

for n odd $\cos n\circledR = -1$ $\qquad \underline{\therefore b_n = \frac{10}{n\circledR}}$

In the Fourier series for $f(x)$, therefore,

$$a_0 = 0, \qquad a_n = 0, \qquad b_n = \frac{10}{n\pi}(-1)^{n+1}.$$

In this particular case the constant term and the coefficients of the cosine terms are all zero. The series therefore consists entirely of sine terms.

$$F(x) = \frac{10}{\pi}\left\{\sin(x) - \frac{1}{2}\sin(2x) + \frac{1}{3}\sin(3x) - \frac{1}{4}\sin(4x) + ...\right\}$$

5. Consider the values between $x = -\pi$ and $x = \pi$. This is an even function so there are no sine terms in the required series.

$$f(x) = 0 \qquad -\pi < x < -\frac{\pi}{2}.$$

$$f(x) = 10 \qquad -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

$$f(x) = 0 \qquad \frac{\pi}{2} < x < \pi.$$

Now $$F(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$F(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right).$$

Where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$

(a) To find a_o

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{-\pi/2} 0\,dx + \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} 10\,dx + \frac{1}{\pi}\int_{\pi/2}^{\pi} 0\,dx$$

$$a_0 = 10.$$

(b) To find a_n

$$a_n = \frac{1}{®}\int_{-®}^{®} f(x)\cos nx\,dx \quad \text{where}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$
$$= \frac{1}{\pi}\int_{-\pi}^{-\pi/2} 0\cos(nx)\,dx + \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} 10\cos(nx)\,dx + \frac{1}{\pi}\int_{\pi/2}^{\pi} 0\cos(nx)\,dx$$

Therefore we have

$$a_n = \frac{10}{\pi}\left[\frac{\sin(nx)}{n}\right]_{-\pi/2}^{\pi/2}$$
$$= \frac{10}{n\pi}\left[\sin\left(\frac{n\pi}{2}\right) - \sin\left(-\frac{n\pi}{2}\right)\right]$$
$$= \frac{10}{n\pi}\left[\sin\left(\frac{n\pi}{2}\right) + \sin\left(\frac{n\pi}{2}\right)\right] = \frac{20}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

For n even $a_n = 0$ and for $n = 1,5,9,\ldots.\ \sin\frac{n®}{2} = 1$. So we have

$$a_n = \frac{20}{n\pi} \quad \text{for } n = 1,5,9$$

Similarly for $n = 3,7,11,\ldots$ $\sin\left(\frac{n\pi}{2}\right) = -1$ we have

$$a_n = -\frac{20}{n\pi}$$

We have

$$F(x) = 5 + \frac{20}{\pi}\cos(x) - \frac{20}{3\pi}\cos(3x) + \frac{20}{5\pi}\cos(5x) - \frac{20}{7\pi}\cos(7x) + \ldots$$

$$\text{or } F(x) = 5 + \frac{20}{\pi}\left(\cos(x) - \frac{1}{3}\cos(3x) + \frac{1}{5}\cos(5x) - \frac{1}{7}\cos(7x) + \ldots..\right)$$

6. Considering the wave form between $x = -®$ and $x = ®$.

(a) To find a_o, $$a_0 = \frac{1}{\circledR}\int_{-\circledR}^{\circledR} f(x)dx = \frac{1}{\circledR}\int_{-\circledR}^{0} 0dx + \frac{1}{\circledR}\int_{0}^{\circledR} xdx$$

$$a_0 = \frac{\circledR}{2}.$$

(b) To find a_n, $$a_n = \frac{1}{\circledR}\int_{-\circledR}^{\circledR} f(x)\cos nx\, dx.$$

$$a_n = \frac{1}{\circledR}\int_{-\circledR}^{\circledR} f(x)\cos nx\, dx = \frac{1}{\circledR}\int_{-\circledR}^{0} 0\cos nx\, dx + \frac{1}{\circledR}\int_{0}^{\circledR} x\cos nx\, dx.$$

$$a_n = \frac{1}{\circledR}\left\{0 + \left[x\frac{\sin nx}{n}\right]_0^{\circledR} - \frac{1}{n}\int_0^{\circledR}\sin nx\, dx\right\}$$

$$a_n = \frac{1}{\circledR}\left\{\frac{\circledR}{n}\sin n\circledR + \frac{1}{n^2}(\cos n\circledR - 1)\right\}$$

For $n = 1,2,3,\ldots\ \sin n\circledR = 0$

For $n = 1,3,5,\ldots\ \cos n\circledR = -1$

For $n = 2,4,6,\ldots\ \cos n\circledR = 1$

$$\therefore a_n = \frac{1}{n^2\circledR}\left\{(-1)^n - 1\right\} = 0 \text{ for } n \text{ even.}$$

$$\therefore a_n = -\frac{2}{n^2\circledR} \text{ for } n \text{ odd.}$$

To find b_n, $$b_n = \frac{1}{\circledR}\int_{-\circledR}^{\circledR} f(x)\sin nx\, dx$$

$$b_n = \frac{1}{\circledR}\int_{-\circledR}^{\circledR} f(x)\sin nxdx = \frac{1}{\circledR}\int_{-\circledR}^{0} 0\sin nxdx + \frac{1}{\circledR}\int_{0}^{\circledR} x\sin nxdx$$

$$b_n = \frac{1}{\circledR}\left\{0 + \left[x\frac{-\cos nx}{n}\right]_0^{\circledR} + \frac{1}{n}\int_0^{\circledR}\cos nx\, dx\right\}$$

$$b_n = \frac{1}{\circledR}\left\{\left[-\frac{\circledR}{n}\cos n\circledR - 0\right] + \frac{1}{n^2}(\sin n\circledR - 0)\right\}$$

For $n = 1,2,3,\ldots\ \sin n\circledR = 0$

For $n = 1,3,5,\ldots\ \cos n\circledR = -1$

For $n = 2,4,6,\ldots.\ \cos n® = 1$

$$\therefore b_n = \frac{1}{®}\left\{\frac{-®}{n}\cos n® + 0\right\}$$

$$\therefore b_n = -\frac{1}{n}(-1)^n = \frac{(-1)^{n+1}}{n}$$

In this case the Fourier series is given by:

$$F(x) = \frac{\pi}{4} - \frac{2}{\pi}\left\{\cos(x) + \frac{1}{3^2}\cos(3x) + \frac{1}{5^2}\cos(5x) + \ldots..\right\} +$$
$$\left\{\sin(x) - \frac{1}{2}\sin(2x) + \frac{1}{3}\sin(3x) - \frac{1}{4}\sin(4x) + \ldots..\right\}$$