Complete Solutions to Exercise I(d)

2. *Proof.* Let $P(n)$ be the given proposition: $2+5+8+...+(3n-1)=\frac{1}{2}n(3n+1)$

Check $P(1)$. Substituting $n=1$ gives

$$2=\frac{1}{2}(1)(3+1) \qquad \surd$$

Hence $P(1)$ is true. Assume the proposition is true for $n=k$:

$$2+5+8+...+(3k-1)=\frac{1}{2}k(3k+1) \qquad (*)$$

Required to prove the result for $n=k+1$. We need to prove

$$\begin{aligned}2+5+8+...+(3k-1)+(3(k+1)-1)&=\frac{1}{2}(k+1)(3(k+1)+1)\\&=\frac{1}{2}(k+1)(3k+4) \qquad (**)\end{aligned}$$

How do we prove (**)?

By examining the Left-hand side and using (*).

$$\begin{aligned}2+5+...+(3k-1)+(3(k+1)-1)&=\underbrace{2+5+8+...+(3k-1)}_{=\frac{1}{2}k(3k+1)\text{ by }(*)}+\underbrace{(3(k+1)-1)}_{=3k+2}\\&=\frac{1}{2}k(3k+1)+(3k+2)\\&=\frac{1}{2}\left[k(3k+1)+2(3k+2)\right] && \left[\text{Rewriting } (3k+2)=\frac{1}{2}2(3k+2)\right]\\&=\frac{1}{2}\left[3k^2+\underbrace{k+6k}_{=7k}+4\right] && [\text{Expanding Brackets}]\\&=\frac{1}{2}\left[3k^2+7k+4\right]\\&=\frac{1}{2}\left[(k+1)(3k+4)\right] && [\text{Factorizing Quadratic}]\end{aligned}$$

The last line is the Right-hand side of (**). Therefore we have shown (**) and by induction we have our given proposition.

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4. *Proof.* Let $P(n)$ be the given proposition: $1^3+2^3+3^3+...+n^3=(1+2+3+4+...+n)^2$

Check $P(1)$. Substituting $n=1$ gives

$$1^3=(1)^2 \qquad \surd$$

Hence $P(1)$ is true. Assume the proposition is true for $n=k$:

$$1^3+2^3+3^3+\ldots+k^3=\left(1+2+3+4+\ldots+k\right)^2$$

Required to prove the proposition for $n=k+1$:

$$1^3+2^3+3^3+\ldots+k^3+\left(k+1\right)^3=\left(1+2+3+4+\ldots+k+\left(k+1\right)\right)^2 \qquad (\dagger)$$

Using the given hint on the Left-hand side of (†) gives

$$1^3+2^3+3^3+\ldots+k^3+\left(k+1\right)^3=\frac{1}{4}\left(k+1\right)^2\left(k+2\right)^2 \qquad (\dagger\dagger)$$

[By Question 3 with $n=k+1$]

How do we show this is equal to the Right-hand side of (†)?

By Example 43 which is

$$1+2+3+4+\ldots+n=\frac{1}{2}n\left(n+1\right)$$

Substituting $n=k+1$ into this we have

$$1+2+3+4+\ldots+\left(k+1\right)=\frac{1}{2}\left(k+1\right)\left(k+2\right)$$

Squaring both sides gives

$$\begin{aligned}\left(1+2+3+4+\ldots+\left(k+1\right)\right)^2&=\left[\frac{1}{2}\left(k+1\right)\left(k+2\right)\right]^2\\&=\frac{1}{4}\left(k+1\right)^2\left(k+2\right)^2\end{aligned}$$

This the same as the Right-hand side of (††). Therefore we have shown (†) which means the result follows by induction.

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10. *Proof.* Let $P(n)$ be the given proposition:

$$1^4+2^4+3^4+\ldots+n^4=\frac{n\left(n+1\right)\left(2n+1\right)\left(3n^2+3n-1\right)}{30}$$

Check $P(1)$. Substituting $n=1$ gives

$$1^4=\frac{1\left(1+1\right)\left(2+1\right)\left(3+3-1\right)}{30}=\frac{1\left(2\right)\left(3\right)\left(5\right)}{30}=\frac{30}{30}=1 \quad \surd$$

Hence $P(1)$ is true. Assume the proposition is true for $n=k$:

$$1^4+2^4+3^4+\ldots+k^4=\frac{k\left(k+1\right)\left(2k+1\right)\left(3k^2+3k-1\right)}{30} \qquad (*)$$

Required to prove the proposition for $n=k+1$:

$$1^4+2^4+3^4+\ldots+k^4+(k+1)^4=\frac{(k+1)((k+1)+1)(2(k+1)+1)\left(3(k+1)^2+3(k+1)-1\right)}{30}$$

$$=\frac{(k+1)(k+2)(2k+3)\left(3\left(k^2+2k+1\right)+3k+3-1\right)}{30}\qquad\left[\begin{array}{l}\text{Simplifying}\\\text{and Expanding}\end{array}\right]$$

$$=\frac{(k+1)(k+2)(2k+3)\left(3k^2+6k+3+3k+2\right)}{30}$$

$$=\frac{(k+1)(k+2)(2k+3)\left(3k^2+9k+5\right)}{30}\qquad(**)$$

Expanding the Left-hand side of (**) using (*) gives

$$1^4+2^4+3^4+\ldots+k^4+(k+1)^4=\underbrace{1^4+2^4+3^4+\ldots+k^4}_{=\frac{k(k+1)(2k+1)\left(3k^2+3k-1\right)}{30}\text{ by (*)}}+(k+1)^4$$

$$=\frac{k(k+1)(2k+1)\left(3k^2+3k-1\right)}{30}+(k+1)^4$$

$$=\frac{(k+1)}{30}\left[k(2k+1)\left(3k^2+3k-1\right)+30(k+1)^3\right]$$

Expanding the square brackets gives:

$$\left[k(2k+1)\left(3k^2+3k-1\right)+30(k+1)^3\right]=\left(2k^2+k\right)\left(3k^2+3k-1\right)+30\left(k^3+3k^2+3k+1\right)$$

$$=6k^4+6k^3-2k^2+3k^3+3k^2-k+30k^3+90k^2+90k+30$$

$$=6k^4+39k^3+91k^2+89k+30$$

Left-hand side of (**) is equal to

$$\frac{(k+1)}{30}\left[k(2k+1)\left(3k^2+3k-1\right)+30(k+1)^3\right]=\frac{(k+1)}{30}\left[6k^4+39k^3+91k^2+89k+30\right]$$

Expanding the Right-hand side of (**) also gives this result:

$$\frac{(k+1)(k+2)(2k+3)\left(3k^2+9k+5\right)}{30}=\frac{(k+1)}{30}\underbrace{\left[(k+2)(2k+3)\left(3k^2+9k+5\right)\right]}_{=6k^4+39k^3+91k^2+89k+30}$$

$$=\frac{(k+1)}{30}\left[6k^4+39k^3+91k^2+89k+30\right]$$

Hence the Left-hand side is equal to the Right-hand side of (**). We have shown $P(k)\ \Rightarrow\ P(k+1)$ therefore our given result follows by induction,

$$1^4+2^4+3^4+\ldots+n^4=\frac{n(n+1)(2n+1)\left(3n^2+3n-1\right)}{30}$$

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11. *Proof.* Let $P(n)$ be the given proposition:

$$1^5+2^5+3^5+\ldots+n^5=\frac{n^2(n+1)^2(2n^2+2n-1)}{12}$$

Check $P(1)$. Substituting $n=1$ gives

$$1^5=\frac{1^2(1+1)^2\left(2(1)^2+2(1)-1\right)}{12}=\frac{2^2(2+2-1)}{12}=\frac{4(3)}{12}=1 \quad \surd$$

Hence $P(1)$ is true. Assume the proposition is true for $n=k$:

$$1^5+2^5+3^5+\ldots+k^5=\frac{k^2(k+1)^2(2k^2+2k-1)}{12} \qquad (€)$$

Required to prove the proposition for $n=k+1$:

$$1^5+2^5+3^5+\ldots+k^5+(k+1)^5=\frac{(k+1)^2\left((k+1)+1\right)^2\left(2(k+1)^2+2(k+1)-1\right)}{12}$$

$$=\frac{(k+1)^2(k+2)^2\left(2(k^2+2k+1)+2k+2-1\right)}{12}$$

$$=\frac{(k+1)^2(k+2)^2(2k^2+4k+2+2k+2-1)}{12}$$

$$=\frac{(k+1)^2(k+2)^2(2k^2+6k+3)}{12} \qquad (!)$$

Expanding the Left-hand side of (!) using (€) gives

$$1^5+2^5+3^5+\ldots+k^5+(k+1)^5=\underbrace{1^5+2^5+3^5+\ldots+k^5}_{=\frac{k^2(k+1)^2(2k^2+2k-1)}{12}}+(k+1)^5$$

$$=\frac{k^2(k+1)^2(2k^2+2k-1)}{12}+(k+1)^5$$

$$=\frac{(k+1)^2}{12}\left[k^2(2k^2+2k-1)+12(k+1)^3\right] \qquad \left[\text{Taking Out a Common Factor of } \frac{(k+1)^2}{12}\right]$$

$$=\frac{(k+1)^2}{12}\left[2k^4+2k^3-k^2+12(k^3+3k^2+3k+1)\right] \qquad [\text{Expanding Brackets}]$$

$$=\frac{(k+1)^2}{12}\left[2k^4+2k^3-k^2+12k^3+36k^2+36k+12\right]$$

$$=\frac{(k+1)^2}{12}\left[2k^4+14k^3+35k^2+36k+12\right] \qquad [\text{Collecting Like Terms}]$$

Expanding the Right-hand side of (!) gives:

$$\frac{(k+1)^2(k+2)^2(2k^2+6k+3)}{12}=\frac{(k+1)^2}{12}\left[(k+2)^2(2k^2+6k+3)\right]$$

$$=\frac{(k+1)^2}{12}\left[(k^2+4k+4)(2k^2+6k+3)\right]$$

$$=\frac{(k+1)^2}{12}\left[2k^4+6k^3+3k^2+8k^3+24k^2+12k+8k^2+24k+12\right]$$

$$\left[\text{Expanding } (k^2+4k+4)(2k^2+6k+3)\right]$$

$$=\frac{(k+1)^2}{12}\left[2k^4+14k^3+35k^2+36k+12\right]$$

Hence the Left-hand side is equal to the Right-hand side of (!). We have shown $P(k) \Rightarrow P(k+1)$ therefore our given result follows by induction,

$$1^5+2^5+3^5+\ldots+n^5=\frac{n^2(n+1)^2(2n^2+2n-1)}{12}$$

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16. *Proof.* We first check the proposition for $n=1$

$$a=\frac{a(1-r)}{1-r}=a \qquad \text{[Cancelling } (1-r)\text{'s]}$$

Hence the proposition is true for $n=1$. *What is our next step?*

Assume the proposition is true for $n=k$, that is

$$a+ar+ar^2+\ldots+ar^{k-1}=\frac{a(1-r^k)}{1-r} \qquad (\$)$$

We need to prove the proposition for $n=k+1$ which is the following;

$$a+ar+ar^2+\ldots+ar^{k-1}+ar^k=\frac{a(1-r^{k+1})}{1-r} \qquad (\#)$$

What do we need to prove?

Left-hand side is equa1 to the Right-hand side of (#). Examining the Left-hand side of (#) and using ($) we have

$$a+ar+ar^2+\ldots+ar^{k-1}+ar^k=\underbrace{a+ar+ar^2+\ldots+ar^{k-1}}_{=\frac{a\left(1-r^k\right)}{1-r}\text{ by (\$)}}+ar^k$$

$$=\frac{a\left(1-r^k\right)}{1-r}+ar^k$$

$$=\frac{a\left(1-r^k\right)+ar^k\left(1-r\right)}{1-r} \qquad \text{[Common Denominator]}$$

$$=\frac{a-ar^k+ar^k-ar^kr}{1-r} \qquad \text{[Expanding Brackets on Numerator]}$$

$$=\frac{a-ar^{k+1}}{1-r} \qquad \left[\text{Because } -ar^k+ar^k=0\right]$$

$$=\frac{a\left(1-r^{k+1}\right)}{1-r} \qquad \text{[Factorizing Numerator]}$$

The last line is the Right-hand side of (#). Therefore we have shown Left-hand side is equa1 to the Right-hand side of (#). Hence we have our result.

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17. *Proof.* By applying mathematical induction we have:

Check the result is true for $n=1$, that is

$$\sin(x)=\frac{\cos\left(\frac{x}{2}\right)-\cos\left(\frac{2(1)+1}{2}x\right)}{2\sin\left(\frac{x}{2}\right)}$$

$$=\frac{\cos\left(\frac{x}{2}\right)-\cos\left(\frac{3}{2}x\right)}{2\sin\left(\frac{x}{2}\right)} \qquad (\dagger)$$

How do we show the Right-hand side simplifies to $\sin(x)$ *?*

We need to use the trigonometric identity:

$$\cos(A)-\cos(B)=-2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

on the numerator of (†).

$$\begin{aligned}\cos\left(\frac{x}{2}\right)-\cos\left(\frac{3x}{2}\right) &= -2\sin\left(\frac{x+3x}{4}\right)\sin\left(\frac{x-3x}{4}\right)\\ &= -2\sin(x)\sin\left(-\frac{x}{2}\right) && \text{[Simplifying]}\\ &= -2\sin(x)\left(-\sin\left(\frac{x}{2}\right)\right) && \left[\text{Because } \sin\left(-\overline{\ }\right) = -\sin\left(\overline{\ }\right)\right]\\ &= 2\sin(x)\sin\left(\frac{x}{2}\right)\end{aligned}$$

Substituting this into (†) gives

$$\sin(x) = \frac{2\sin(x)\sin\left(\frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)} = \sin(x) \qquad \left[\text{Cancelling } 2\sin\left(\frac{x}{2}\right)\right]$$

Hence the proposition is true for $n=1$. Next we assume the proposition is true for $n=k$:

$$\sin(x)+\sin(2x)+\ldots+\sin(kx) = \frac{\cos\left(\frac{x}{2}\right)-\cos\left(\frac{2k+1}{2}x\right)}{2\sin\left(\frac{x}{2}\right)} \qquad (*)$$

We need to prove the proposition for $n=k+1$, that is

$$\begin{aligned}\sin(x)+\sin(2x)+\ldots+\sin(kx)+\sin\left(\left(k+1\right)x\right) &= \frac{\cos\left(\frac{x}{2}\right)-\cos\left(\frac{2\left(k+1\right)+1}{2}x\right)}{2\sin\left(\frac{x}{2}\right)}\\ &= \frac{\cos\left(\frac{x}{2}\right)-\cos\left(\left(\frac{2k+3}{2}\right)x\right)}{2\sin\left(\frac{x}{2}\right)} \qquad (**)\end{aligned}$$

Wha

t do we need to show?

The Left-hand side is equal to the Right-hand side of (**). Let's examine the Left-hand side first.

$$\underbrace{\sin(x)+...+\sin(kx)}_{=\frac{\cos\left(\frac{x}{2}\right)-\cos\left(\frac{2k+1}{2}x\right)}{2\sin\left(\frac{x}{2}\right)} \text{ by } (*)}+\sin\left(\left(k+1\right)x\right)=\frac{\cos\left(\frac{x}{2}\right)-\cos\left(\frac{2k+1}{2}x\right)}{2\sin\left(\frac{x}{2}\right)}+\sin\left(\left(k+1\right)x\right)$$

$$=\frac{\cos\left(\frac{x}{2}\right)-\cos\left(\frac{2k+1}{2}x\right)+2\sin\left(\frac{x}{2}\right)\sin\left(\left(k+1\right)x\right)}{2\sin\left(\frac{x}{2}\right)} \quad \left[\text{Common Denominator}\right]$$

What do we do next?

We can use the following trigonometric identity on the last term of the numerator:

$$2\sin\left(A\right)\sin\left(B\right)=\cos\left(A-B\right)-\cos\left(A+B\right)$$

$$2\sin\left(\frac{x}{2}\right)\sin\left(\left(k+1\right)x\right)=\left[\cos\left(\frac{x}{2}-\left(k+1\right)x\right)-\cos\left(\frac{x}{2}+\left(k+1\right)x\right)\right]$$

$$=\left[\cos\left(\frac{x}{2}-\frac{\left(2k+2\right)x}{2}\right)-\cos\left(\frac{x}{2}+\frac{\left(2k+2\right)x}{2}\right)\right]$$

$$=\left[\cos\left(\frac{x-2kx-2x}{2}\right)-\cos\left(\frac{x+2kx+2x}{2}\right)\right]$$

$$=\left[\cos\left(\frac{-x-2kx}{2}\right)-\cos\left(\frac{3x+2kx}{2}\right)\right]$$

$$=\left[\cos\left(\frac{x+2kx}{2}\right)-\cos\left(\frac{3x+2kx}{2}\right)\right] \quad \left[\text{Using } \cos\left(-\theta\right)=\cos\left(\theta\right)\right]$$

$$=\left[\cos\left(\frac{\left(2k+1\right)x}{2}\right)-\cos\left(\frac{\left(2k+3\right)x}{2}\right)\right]$$

Substituting this into the above we have

$$\sin(x)+...+\sin\left(\left(k+1\right)x\right)=\frac{\cos\left(\frac{x}{2}\right)-\cos\left(\frac{2k+1}{2}x\right)+\left[\cos\left(\frac{\left(2k+1\right)x}{2}\right)-\cos\left(\frac{\left(2k+3\right)x}{2}\right)\right]}{2\sin\left(\frac{x}{2}\right)}$$

$$=\frac{\cos\left(\frac{x}{2}\right)-\cos\left(\frac{\left(2k+3\right)x}{2}\right)}{2\sin\left(\frac{x}{2}\right)}$$

$\left[\text{Because } -\cos\left(\frac{2k+1}{2}x\right)+\cos\left(\frac{(2k+1)x}{2}\right)=0\right]$. Hence, we have the Right-hand side of (**). Therefore we have our required result and the proposition is proved by induction.

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18. *Proof.* We first check the proposition for $n=1$

$$(a+b)^1 = a^1 + b^1 = a+b$$

Hence the proposition is true for $n=1$. *What is our next step?*

Assume the proposition is true for $n=k$, that is

$$(a+b)^k = a^k + ka^{k-1}b + \frac{k(k-1)}{2!}a^{k-2}b^2 + \ldots + b^k \qquad (*)$$

We need to prove the proposition for $n=k+1$ which is the following;

$$\begin{aligned}(a+b)^{k+1} &= a^{k+1} + (k+1)a^{k-1+1}b + \frac{(k+1)\left((k+1)-1\right)}{2!}a^{(k+1)-2}b^2 + \ldots + b^{k+1} \\ &= a^{k+1} + (k+1)a^k b + \frac{(k+1)k}{2!}a^{k-1}b^2 + \ldots + b^{k+1}\end{aligned}$$

What do we need to show to prove this?

Left-hand side is equa1 to the Right-hand side. *How?*

Using (*) and algebraic manipulation.

$$
\begin{aligned}
\left(a+b\right)^{k+1} &= \left(a+b\right)^{k}\left(a+b\right)^{1} \\
&= \left(\underbrace{a^{k} + ka^{k-1}b + \frac{k\left(k-1\right)}{2!}a^{k-2}b^{2} + \ldots + b^{k}}_{\text{by (*)}}\right)\left(a+b\right) \\
&= \underbrace{a^{k}a + ka^{k-1}ba + \frac{k\left(k-1\right)}{2!}a^{k-2}b^{2}a + \ldots + b^{k}a}_{\text{Multiplying the Long Bracket by } a} + \\
&\qquad \underbrace{a^{k}b + ka^{k-1}bb + \frac{k\left(k-1\right)}{2!}a^{k-2}b^{2}b + \ldots + b^{k}b}_{\text{Multiplying the Long Bracket by } b} \\
&= a^{k+1} + ka^{k}b + \frac{k\left(k-1\right)}{2!}a^{k-1}b^{2} + \ldots + ab^{k} + \\
&\qquad a^{k}b + ka^{k-1}b^{2} + \frac{k\left(k-1\right)}{2!}a^{k-2}b^{3} + \ldots + b^{k+1} \\
&\qquad \left[\text{Simplifying by using rules of Indices}\right] \\
&= a^{k+1} + \left(k+1\right)a^{k}b + \left[\frac{k\left(k-1\right)}{2!} + k\right]a^{k-1}b^{2} + \ldots + b^{k+1} \qquad \left[\begin{matrix}\text{Collecting} \\ \text{like Terms}\end{matrix}\right] \\
&= a^{k+1} + \left(k+1\right)a^{k}b + \underbrace{\left[\frac{k\left(k+1\right)}{2!}\right]}_{\text{because } \frac{k(k-1)}{2!}+k=\frac{k(k+1)}{2!}} a^{k-1}b^{2} + \ldots + b^{k+1}
\end{aligned}
$$

Hence we have

$$
\left(a+b\right)^{k+1} = a^{k+1} + \left(k+1\right)a^{k}b + \frac{\left(k+1\right)k}{2!}a^{k-1}b^{2} + \ldots + b^{k+1}
$$

The required result. We have proven the binomial theorem for all natural numbers.