

## Complete Solutions to Exercise 1g

1. We need to prove the sum of all the even numbers is equal to  $n(n+1)$ :

$$2 + 4 + 6 + \dots + 2n = n(n+1)$$

*Proof.*

We use mathematical induction. Let  $P(n)$  be the above proposition. We first check that  $P(1)$  is true:

$$2 = 1(1+1)$$

We assume the result is true for  $n = k$ :

$$2 + 4 + 6 + \dots + 2k = k(k+1) \quad (*)$$

Required to prove the proposition for  $n = k+1$ , that is we need to prove

$$2 + 4 + 6 + \dots + 2k + 2(k+1) = (k+1)(k+2)$$

Separating out the LHS of this last equation into the first  $2k$  terms and  $2(k+1)$  term:

$$\begin{aligned} \underbrace{2 + 4 + 6 + \dots + 2k}_{=k(k+1) \text{ by } (*)} + 2(k+1) &= k(k+1) + 2(k+1) \\ &= (k+1)(k+2) \quad [\text{Factorizing}] \end{aligned}$$

This completes our proof.

2. Using mathematical induction to prove

$$(1 \times 2) + (2 \times 3) + (3 \times 4) + \dots + n(n+1) = \frac{1}{3}n(n+1)(n+2)$$

*Proof.*

Clearly the result is true for  $n = 1$ :

$$(1 \times 2) = \frac{1}{3}1(1+1)(1+2)$$

Assume the following is true:

$$(1 \times 2) + (2 \times 3) + (3 \times 4) + \dots + k(k+1) = \frac{1}{3}k(k+1)(k+2) \quad (\dagger)$$

Required to prove that

$$(1 \times 2) + (2 \times 3) + \dots + k(k+1) + (k+1)(k+2) = \frac{1}{3}(k+1)(k+2)(k+3)$$

Again splitting the sum on the LHS:

$$\begin{aligned} \underbrace{(1 \times 2) + (2 \times 3) + \dots + k(k+1)}_{=\frac{1}{3}k(k+1)(k+2) \text{ by } (\dagger)} + (k+1)(k+2) &= \frac{1}{3}k(k+1)(k+2) + \frac{1}{3}3(k+1)(k+2) \\ &= \frac{1}{3}(k+1)(k+2)[k+3] \end{aligned}$$

This completes our proof.

3. Very similar to solution 2.
4. We need to prove the proposition  $P(n)$  given by

$$1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$$

*Proof.*

Checking  $P(1)$ . (Actually we could check  $P(0)$ .):

$$1 + 2 = 2^{1+1} - 1$$

Assume the result is true for  $P(k)$ :

$$1 + 2 + 2^2 + \cdots + 2^k = 2^{k+1} - 1 \quad (\dagger)$$

Required to prove

$$1 + 2 + 2^2 + \cdots + 2^k + 2^{k+1} = 2^{k+2} - 1 \quad (\dagger\dagger)$$

Using  $(\dagger)$  on the LHS for the sum up to  $2^k$ :

$$\underbrace{1 + 2 + 2^2 + \cdots + 2^k}_{=2^{k+1}-1} + 2^{k+1} = 2^{k+1} - 1 + 2^{k+1} = 2(2^{k+1}) - 1 = 2^{k+2} - 1$$

Hence by mathematical induction we have our result.

5. This is really the same proposition as question 4. Since we have proven the proposition in question 4 so move the 1 to the right gives us  $2 + 2^2 + \cdots + 2^n = 2^{n+1} - 2$ .
6. *Proof.* Let  $P(n)$  be the given proposition:  $2 + 5 + 8 + \cdots + (3n - 1) = \frac{1}{2}n(3n + 1)$ .

Check  $P(1)$ . Substituting  $n = 1$  gives

$$2 = \frac{1}{2}(1)(3 + 1)$$

Hence  $P(1)$  is true. Assume the proposition is true for  $n = k$ :

$$2 + 5 + 8 + \cdots + (3k - 1) = \frac{1}{2}k(3k + 1) \quad (*)$$

Required to prove the result for  $n = k + 1$ . We need to prove

$$\begin{aligned} 2 + 5 + 8 + \cdots + (3k - 1) + (3(k + 1) - 1) &= \frac{1}{2}(k + 1)(3(k + 1) + 1) \\ &= \frac{1}{2}(k + 1)(3k + 4) \end{aligned} \quad (**)$$

*How do we prove (\*\*)?*

By examining the Left Hand Side and using (\*).

$$\begin{aligned}
2+5+\dots+(3k-1)+(3(k+1)-1) &= \underbrace{2+5+8+\dots+(3k-1)}_{=\frac{1}{2}k(3k+1) \text{ by } (**)} + \underbrace{(3(k+1)-1)}_{=3k+2} \\
&= \frac{1}{2}k(3k+1) + (3k+2) \\
&= \frac{1}{2}[k(3k+1) + 2(3k+2)] && \left[ \text{Rewriting } (3k+2) = \frac{1}{2}2(3k+2) \right] \\
&= \frac{1}{2}\left[3k^2 + \underbrace{k+6k}_{=7k} + 4\right] && \left[ \text{Expanding Brackets} \right] \\
&= \frac{1}{2}[3k^2 + 7k + 4] \\
&= \frac{1}{2}[(k+1)(3k+4)] && \left[ \text{Factorizing Quadratic} \right]
\end{aligned}$$

The last line is the Right Hand Side of (\*\*). Therefore we have shown (\*\*) and by induction we have our given proposition.

7. We need to prove that

$$1^3 + 2^3 + 3^3 + \dots + n^3 = \frac{1}{4}n^2(n+1)^2$$

*Proof.*

Check  $P(1)$ :

$$1^3 = \frac{1}{4}1^2(1+1)^2$$

Assume the result is true for  $n = k$ :

$$1^3 + 2^3 + 3^3 + \dots + k^3 = \frac{1}{4}k^2(k+1)^2 \quad (\dagger)$$

Required to prove

$$1^3 + 2^3 + 3^3 + \dots + k^3 + (k+1)^3 = \frac{1}{4}(k+1)^2(k+2)^2$$

Separating the sum of the LHS and using ( $\dagger$ ):

$$\begin{aligned}
\underbrace{1^3 + 2^3 + 3^3 + \dots + k^3}_{=\frac{1}{4}k^2(k+1)^2 \text{ by } (\dagger)} + (k+1)^3 &= \frac{1}{4}k^2(k+1)^2 + (k+1)^3 \\
&= \frac{1}{4}(k+1)^2[k^2 + 4(k+1)] && \left[ \text{Factorizing and writing } (k+1)^3 = \frac{1}{4}(k+1)^2 4(k+1) \right] \\
&= \frac{1}{4}(k+1)^2[k^2 + 4k + 4] = \frac{1}{4}(k+1)^2[k+2]^2
\end{aligned}$$

Hence this is our required result.

8. *Proof.* Let  $P(n)$  be the given proposition:  $1^3 + 2^3 + 3^3 + \dots + n^3 = (1+2+3+4+\dots+n)^2$

Check  $P(1)$ . Substituting  $n=1$  gives

$$1^3 = (1)^2$$

Hence  $P(1)$  is true. Assume the proposition is true for  $n = k$  :

$$1^3 + 2^3 + 3^3 + \dots + k^3 = (1 + 2 + 3 + 4 + \dots + k)^2$$

Required to prove the proposition for  $n = k + 1$  :

$$1^3 + 2^3 + 3^3 + \dots + k^3 + (k+1)^3 = (1 + 2 + 3 + 4 + \dots + k + (k+1))^2 \quad (\dagger)$$

Using the given hint on the Left Hand Side of  $(\dagger)$  gives

$$1^3 + 2^3 + 3^3 + \dots + k^3 + (k+1)^3 = \frac{1}{4}(k+1)^2(k+2)^2 \quad (\dagger\dagger)$$

[By result of Question 3 with  $n = k + 1$ ]

How do we show this is equal to the Right Hand Side of  $(\dagger)$ ?

By Example 1 which is

$$1 + 2 + 3 + 4 + \dots + n = \frac{1}{2}n(n+1)$$

Substituting  $n = k + 1$  into this we have

$$1 + 2 + 3 + 4 + \dots + (k+1) = \frac{1}{2}(k+1)(k+2)$$

Squaring both sides gives

$$\begin{aligned} (1 + 2 + 3 + 4 + \dots + (k+1))^2 &= \left[ \frac{1}{2}(k+1)(k+2) \right]^2 \\ &= \frac{1}{4}(k+1)^2(k+2)^2 \end{aligned}$$

This the same as the Right Hand Side of  $(\dagger\dagger)$ . Therefore we have shown  $(\dagger)$  which means the result follows by induction.

9. This is the sum of the finite geometric series:

$$1 + r + r^2 + \dots + r^n = \frac{1 - r^{n+1}}{1 - r} \quad (r \neq 1)$$

*Proof.*

Check the result for  $n = 1$  :

$$1 + r = \frac{1 - r^{1+1}}{1 - r} = \frac{1 - r^2}{1 - r} = \frac{(1 - r)(1 + r)}{(1 - r)} \stackrel{\text{Cancelling}}{=} 1 + r$$

Hence the result holds for  $n = 1$ . Assume it is true for  $n = k$  :

$$1 + r + r^2 + \dots + r^k = \frac{1 - r^{k+1}}{1 - r} \quad (*)$$

Required to prove the result for  $n = k + 1$  :

$$1 + r + r^2 + \dots + r^k + r^{k+1} = \frac{1 - r^{k+2}}{1 - r} \quad (**)$$

Separating out the LHS sum:

$$\begin{aligned} \underbrace{1+r+r^2+\cdots+r^k}_{=\frac{1-r^{k+1}}{1-r} \text{ by (*)}} + r^{k+1} &= \frac{1-r^{k+1}}{1-r} + r^{k+1} \\ &= \frac{1-r^{k+1} + r^{k+1}(1-r)}{1-r} = \frac{1-r^{k+2}}{1-r} \end{aligned}$$

Hence we have the RHS of (\*\*), so this completes our proof. By mathematical induction we have the given result.

10. The first error is in the first line because  $P(1)$  is not true.

11. The error is in the last line:

$$p + 2k = \text{prime}$$

For example take  $k = p$  then  $p + 2k = p + 2p = 3p$  which is a multiple of 3 greater than 2 so cannot be prime.

12. Let  $P(n)$  be the given proposition:

$$1^3 + 3^3 + 5^3 + \cdots + (2n-1)^3 = n^2(2n^2 - 1)$$

Check  $P(1)$ :

$$1^3 = 1^2(2(1)^2 - 1)$$

Hence the given proposition holds for  $n = 1$ . Assume the following

$$1^3 + 3^3 + 5^3 + \cdots + (2k-1)^3 = k^2(2k^2 - 1) \quad (*)$$

Required to prove

$$1^3 + 3^3 + \cdots + (2k-1)^3 + (2k+1)^3 = (k+1)^2(2(k+1)^2 - 1) = (k+1)^2(2k^2 + 4k + 1)$$

Expanding out the RHS gives

$$(k+1)^2(2k^2 + 4k + 1) = 2k^4 + 8k^3 + 11k^2 + 6k + 1 \quad (**)$$

Separating out the sum on the LHS and using (\*):

$$\underbrace{1^3 + 3^3 + \cdots + (2k-1)^3}_{=k^2(2k^2-1)} + (2k+1)^3 = k^2(2k^2 - 1) + (2k+1)^3$$

Expanding out the RHS of this last equation we have

$$k^2(2k^2 - 1) + (2k+1)^3 = 2k^4 + 8k^3 + 11k^2 + 6k + 1$$

This is identical to the RHS of (\*\*) which means we have our required result.

13. *Proof.* Let  $P(n)$  be the given proposition:

$$1^4 + 2^4 + 3^4 + \cdots + n^4 = \frac{n(n+1)(2n+1)(3n^2 + 3n - 1)}{30}$$

Check  $P(1)$ . Substituting  $n = 1$  gives

$$1^4 = \frac{1(1+1)(2+1)(3+3-1)}{30} = \frac{1(2)(3)(5)}{30} = \frac{30}{30} = 1$$

Hence  $P(1)$  is true. Assume the proposition is true for  $n = k$ :

$$1^4 + 2^4 + 3^4 + \dots + k^4 = \frac{k(k+1)(2k+1)(3k^2+3k-1)}{30} \quad (*)$$

Required to prove the proposition for  $n = k + 1$ :

$$\begin{aligned} 1^4 + 2^4 + 3^4 + \dots + k^4 + (k+1)^4 &= \frac{(k+1)((k+1)+1)(2(k+1)+1)(3(k+1)^2+3(k+1)-1)}{30} \\ &= \frac{(k+1)(k+2)(2k+3)(3(k^2+2k+1)+3k+3-1)}{30} \quad \left[ \begin{array}{l} \text{Simplifying} \\ \text{and Expanding} \end{array} \right] \\ &= \frac{(k+1)(k+2)(2k+3)(3k^2+9k+5)}{30} \quad (**)$$

Expanding the Left Hand Side of (\*\*) using (\*) gives

$$\begin{aligned} 1^4 + 2^4 + 3^4 + \dots + k^4 + (k+1)^4 &= \underbrace{1^4 + 2^4 + 3^4 + \dots + k^4}_{= \frac{k(k+1)(2k+1)(3k^2+3k-1)}{30} \text{ by } (*)} + (k+1)^4 \\ &= \frac{k(k+1)(2k+1)(3k^2+3k-1)}{30} + (k+1)^4 \\ &= \frac{(k+1)}{30} \left[ k(2k+1)(3k^2+3k-1) + 30(k+1)^3 \right] \end{aligned}$$

Expanding the square brackets gives:

$$\begin{aligned} \left[ k(2k+1)(3k^2+3k-1) + 30(k+1)^3 \right] &= (2k^2+k)(3k^2+3k-1) + 30(k^3+3k^2+3k+1) \\ &= 6k^4 + 6k^3 - 2k^2 + 3k^3 + 3k^2 - k + 30k^3 + 90k^2 + 90k + 30 \\ &= 6k^4 + 39k^3 + 91k^2 + 89k + 30 \end{aligned}$$

Left Hand Side of (\*\*) is equal to

$$\frac{(k+1)}{30} \left[ k(2k+1)(3k^2+3k-1) + 30(k+1)^3 \right] = \frac{(k+1)}{30} \left[ 6k^4 + 39k^3 + 91k^2 + 89k + 30 \right]$$

Expanding the Right Hand Side of (\*\*) also gives this result:

$$\begin{aligned} \frac{(k+1)(k+2)(2k+3)(3k^2+9k+5)}{30} &= \frac{(k+1)}{30} \left[ \underbrace{(k+2)(2k+3)(3k^2+9k+5)}_{=6k^4+39k^3+91k^2+89k+30} \right] \\ &= \frac{(k+1)}{30} \left[ 6k^4 + 39k^3 + 91k^2 + 89k + 30 \right] \end{aligned}$$

Hence the Left Hand Side is equal to the Right Hand Side of (\*\*). We have shown  $P(k) \Rightarrow P(k+1)$  therefore our given result follows by induction,

$$1^4 + 2^4 + 3^4 + \dots + n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

14. *Proof.* Let  $P(n)$  be the given proposition:

$$1^5 + 2^5 + 3^5 + \dots + n^5 = \frac{n^2(n+1)^2(2n^2+2n-1)}{12}$$

Check  $P(1)$ . Substituting  $n=1$  gives

$$1^5 = \frac{1^2(1+1)^2(2(1)^2+2(1)-1)}{12} = \frac{2^2(2+2-1)}{12} = \frac{4(3)}{12} = 1$$

Hence  $P(1)$  is true. Assume the proposition is true for  $n=k$ :

$$1^5 + 2^5 + 3^5 + \dots + k^5 = \frac{k^2(k+1)^2(2k^2+2k-1)}{12} \quad (\ominus)$$

Required to prove the proposition for  $n=k+1$ :

$$\begin{aligned} 1^5 + 2^5 + 3^5 + \dots + k^5 + (k+1)^5 &= \frac{(k+1)^2((k+1)+1)^2(2(k+1)^2+2(k+1)-1)}{12} \\ &= \frac{(k+1)^2(k+2)^2(2(k^2+2k+1)+2k+2-1)}{12} \\ &= \frac{(k+1)^2(k+2)^2(2k^2+4k+2+2k+2-1)}{12} \\ &= \frac{(k+1)^2(k+2)^2(2k^2+6k+3)}{12} \quad (!) \end{aligned}$$

Expanding the Left Hand Side of (!) using ( $\ominus$ ) gives

$$\begin{aligned} 1^5 + 2^5 + 3^5 + \dots + k^5 + (k+1)^5 &= \underbrace{1^5 + 2^5 + 3^5 + \dots + k^5}_{\frac{k^2(k+1)^2(2k^2+2k-1)}{12}} + (k+1)^5 \\ &= \frac{k^2(k+1)^2(2k^2+2k-1)}{12} + (k+1)^5 \\ &= \frac{(k+1)^2}{12} \left[ k^2(2k^2+2k-1) + 12(k+1)^3 \right] \quad \left[ \begin{array}{l} \text{Taking Out a Common} \\ \text{Factor of } \frac{(k+1)^2}{12} \end{array} \right] \\ &= \frac{(k+1)^2}{12} \left[ 2k^4 + 2k^3 - k^2 + 12(k^3 + 3k^2 + 3k + 1) \right] \quad [\text{Expanding Brackets}] \\ &= \frac{(k+1)^2}{12} \left[ 2k^4 + 2k^3 - k^2 + 12k^3 + 36k^2 + 36k + 12 \right] \\ &= \frac{(k+1)^2}{12} \left[ 2k^4 + 14k^3 + 35k^2 + 36k + 12 \right] \quad \left[ \begin{array}{l} \text{Collecting Like} \\ \text{Terms} \end{array} \right] \end{aligned}$$

Expanding the Right Hand Side of (!) gives:

$$\begin{aligned}
\frac{(k+1)^2(k+2)^2(2k^2+6k+3)}{12} &= \frac{(k+1)^2}{12} \left[ (k+2)^2(2k^2+6k+3) \right] \\
&= \frac{(k+1)^2}{12} \left[ (k^2+4k+4)(2k^2+6k+3) \right] \\
&= \frac{(k+1)^2}{12} \left[ 2k^4+6k^3+3k^2+8k^3+24k^2+12k+8k^2+24k+12 \right] \\
&\qquad\qquad\qquad \left[ \text{Expanding } (k^2+4k+4)(2k^2+6k+3) \right] \\
&= \frac{(k+1)^2}{12} \left[ 2k^4+14k^3+35k^2+36k+12 \right]
\end{aligned}$$

Hence the Left Hand Side is equal to the Right Hand Side of (!). We have shown  $P(k) \Rightarrow P(k+1)$  therefore our given result follows by induction,

$$1^5 + 2^5 + 3^5 + \cdots + n^5 = \frac{n^2(n+1)^2(2n^2+2n-1)}{12}$$