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Tough Nut to Crack – Integration III: Solution

$$Show\ that\colon \int_0^{2L} \sin\left(\frac{m\pi x}{L}\right) cos\left(\frac{n\pi x}{L}\right) dx = 0$$

$$where\ m\ and\ n\ are\ integers, such\ that\ m \neq n$$

Solution by Samuel Richards

$$let\ u = \frac{\pi x}{L}\ \therefore\ \frac{du}{dx} = \frac{\pi}{L}\ \therefore dx = du\frac{L}{\pi}$$

New Limits:

$$when\ x = 0, u = 0$$

$$when\ x = 2L, u = 2\pi$$

Therefore:

$$\int_0^{2L} \sin\left(\frac{m\pi x}{L}\right) cos\left(\frac{n\pi x}{L}\right) dx = \frac{L}{\pi}\int_0^{2\pi} \sin(mu) cos(nu)\, du = \frac{L}{2\pi}\left(\int_0^{2\pi} \sin((m+n)u) du + \int_0^{2\pi} \sin((m-n)u)\, du\right)$$

$$= \frac{L}{2\pi}\left[\left(\frac{-\cos((m+n)u)}{m+n}\right) + \left(\frac{-\cos((m-n)u)}{m-n}\right)\right]_0^{2\pi}$$

$$= \frac{L}{2\pi}\left[\left(\frac{-\cos((m+n)2\pi)}{m+n}\right) + \left(\frac{-\cos((m-n)2\pi)}{m-n}\right) - \left(\frac{-\cos((m+n)0)}{m+n}\right) - \left(\frac{-\cos((m-n)0)}{m-n}\right)\right]$$

Selecting Variables:

$$As\ m\ and\ n\ are\ integers, let\ (m+n) and\ (m-n) be\ p\ and\ q\ respectively$$

$$\therefore \frac{L}{2\pi}\left[\left(\frac{-\cos((m+n)2\pi)}{m+n}\right) + \left(\frac{-\cos((m-n)2\pi)}{m-n}\right) - \left(\frac{-\cos((m+n)0)}{m+n}\right) - \left(\frac{-\cos((m-n)0)}{m-n}\right)\right]$$

$$= \frac{L}{2\pi}\left[\left(\frac{-\cos(p.2\pi)}{p}\right) + \left(\frac{-\cos(q.2\pi)}{q}\right) - \left(\frac{-\cos(p.0)}{p}\right) - \left(\frac{-\cos(q.0)}{q}\right)\right]$$

$$As\ \cos(p.2\pi) = \cos(q.2\pi) = \cos(p.0) = \cos(q.0) = 1$$

$$\therefore \frac{L}{2\pi}\left[\left(\frac{-\cos(p.2\pi)}{p}\right) + \left(\frac{-\cos(q.2\pi)}{q}\right) - \left(\frac{-\cos(p.0)}{p}\right) - \left(\frac{-\cos(q.0)}{q}\right)\right] = \frac{L}{2\pi}\left[\left(\frac{-1}{p}\right) + \left(\frac{-1}{q}\right) - \left(\frac{-1}{p}\right) - \left(\frac{-1}{q}\right)\right]$$

$$= \frac{L}{2\pi}\left[\left(\frac{1}{p} - \frac{1}{p}\right) + \left(\frac{1}{q} - \frac{1}{q}\right)\right] = \frac{L}{2\pi}[0] = 0$$

Therefore:

$$\int_0^{2L} \sin\left(\frac{m\pi x}{L}\right) cos\left(\frac{n\pi x}{L}\right) dx = 0$$

Ergo, as the inner products of the integral equal zero, they are *orthogonal*.