Solutions to Exercise 2(a)

1. (a) $\sum_{n=1}^{\infty} \sqrt{n}$

(b) $2+4+6+8+\cdots$ is the sum of even numbers. This can be written as $\sum_{n=1}^{\infty} 2n$

(c) $1+3+5+7+\cdots$ is - - — odd numbers. We can write this as $\sum_{n=1}^{\infty} (2n-1)$

(d) This is an alternating series:

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

(e) What do you notice about each term in

$$1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\cdots ?$$

The denominator is 3^n. Hence

$$1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots = \sum_{n=0}^{\infty} \frac{1}{(3)^n}$$

(f) What is the pattern in the sequence

$$\frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \frac{16}{81}, \ldots ?$$

Each term is $\frac{2}{3}$ the previous term. We can write

$$\frac{2}{3}+\frac{4}{9}+\frac{8}{27}+\frac{16}{81}+\cdots = \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$$

2. (a) $\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n$ is a geometric series with ratio $r=\frac{1}{3}$. Using

$$S_\infty = \frac{a}{1-r} \quad \text{provided } |r|<1$$

we have

$$\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n = \frac{1/3}{1-1/3} = \frac{1/3}{2/3} = \frac{1}{2}$$

(b) Similarly we have with common ratio $r=1/4$:

$$\sum_{n=1}^{\infty}\left(\frac{1}{4}\right)^n = \frac{1/4}{1-1/4} = \frac{1/4}{3/4} = \frac{1}{3}$$

(c) We can rewrite $\frac{1}{\pi^n}$ as $\left(\frac{1}{\pi}\right)^n$. Hence

$$\sum_{n=1}^{\infty}\left(\frac{1}{\pi}\right)^n = \frac{1/\pi}{1-1/\pi} = \frac{1}{\pi-1}$$

(d) Similarly

$$\sum_{n=1}^{\infty}\left(\frac{1}{m}\right)^n = \frac{1}{m} + \left(\frac{1}{m}\right)^2 + \left(\frac{1}{m}\right)^3 + \left(\frac{1}{m}\right)^4 + \dots$$

$$= \frac{1/m}{1-1/m} \quad \text{because } \frac{1}{m}<1$$

$$= \frac{1}{m-1}$$

3.(a) Expanding out the given series:

$$\sum_{n=1}^{\infty}\left(\frac{1}{2^{2n-1}}\right) = \frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^5} + \frac{1}{2^7} + \cdots$$

This is a geometric series with $a = 1/2$ & common ratio $r = \frac{1}{2^2} = \frac{1}{4}$.

Since $|r| = \frac{1}{4} < 1$ we use

$$S_\infty = \frac{a}{1-r}$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{2^{2n-1}}\right) = \frac{1/2}{1-1/4} = \frac{1/2}{3/4} = \frac{2}{3}$$

(b) Writing out the given series:

$$\sum_{n=1}^{\infty}\left(\frac{3}{2}\right)^n = \frac{3}{2} + \left(\frac{3}{2}\right)^2 + \left(\frac{3}{2}\right)^3 + \cdots$$

This is a geometric series with ratio $r = \frac{3}{2} > 1$. This series diverges.

(c) Similarly

$$\sum_{n=1}^{\infty} e^n = e + e^2 + e^3 + e^4 + \cdots$$

Geometric series with $r = e > 1$. Series diverges.

(d) Expanding the given series

$$\sum_{n=1}^{\infty} 10\left(\frac{1}{3}\right)^n = 10\left(\frac{1}{3}\right) + 10\left(\frac{1}{3}\right)^2 + 10\left(\frac{1}{3}\right)^3 + \cdots$$

This is a geometric series with $a = \frac{10}{3}$ & $r = \frac{1}{3} < 1$.

Applying the formula

$$S_\infty = \frac{a}{1-r} \quad \text{provided } |r| < 1$$

we have

$$\sum_{n=1}^{\infty} 10\left(\frac{1}{3}\right)^n = \frac{10/3}{1-1/3} = \frac{10/3}{2/3} = 5$$

4. This time we have not been given a formula. In each case we need to write the formula down first.

(a) Dividing the first two terms $\frac{\cancel{8}4}{8} = 1/2$. The common ratio is $r = \frac{1}{2} < 1$ and the first term is 8. Hence

$$8+4+2+1+\cdots = \frac{8}{1-1/2} = 16$$

(b) In the series $3+6+12+24+\cdots$ each term is double the preceding term. We can write the series as

$$3+6+12+24+\cdots = \sum_{n=0}^{\infty} 3(2)^n$$

$\sum_{n=0}^{\infty} 3(2)^n$ is a geometric series with the first term $a=3$ & common ratio $r=2$. Because $|r| = 2 \geq 1$ therefore by (2.3) the series diverges.

(c) ~~F~~ In the series

$$16+12+9+\frac{27}{4}+\cdots$$

each term is $3/4$ the preceding term $\left(\cancel{\frac{16}{12} = 3} \ \left(\frac{12}{16} = \frac{3}{4}\right)\right)$.

We have

$$16+12+9+\frac{27}{4}+\cdots = \sum_{n=0}^{\infty} 16\left(\frac{3}{4}\right)^n$$

This is a geometric series with first term $a=16$ & $r = \frac{3}{4} < 1$.

$$16+12+9+\frac{27}{4}+\cdots = \frac{16}{1-3/4} = \frac{16}{1/4} = 64$$

5(a) The given series

$$\sum_{n=1}^{\infty} \frac{1}{x^n} = \sum_{n=1}^{\infty} \left(\frac{1}{x}\right)^n$$

which is a geometric series with $a = \frac{1}{x}$ & $r = \frac{1}{x}$. Since

$$|r| = \left|\frac{1}{x}\right| = \frac{1}{|x|} < 1 \quad \text{because } |x| > 1$$

the series converges. Using

$$S_\infty = \frac{a}{1-r}$$

we have

$$\sum_{n=1}^{\infty} \frac{1}{x^n} = \frac{1/x}{1 - 1/x} = \frac{1}{x-1}$$ (Multiplying numerator & denominator by x)

(b) We have

$$\sum_{n=1}^{\infty} \left(\frac{x^n}{2^n}\right) = \sum_{n=1}^{\infty} \left(\frac{x}{2}\right)^n$$

This is a geometric series with $a = \frac{x}{2}$ & $r = \frac{x}{2}$. Since

$$|r| = \frac{|x|}{2} < \frac{2}{2} = 1$$

the series converges. Using $S_\infty = \frac{a}{1-r}$ we have

$$\sum_{n=1}^{\infty} \frac{x^n}{2^n} = \frac{x/2}{1 - x/2} = \frac{x}{2-x}$$ (Multiplying top & bottom by 2.)

(c) Expanding the given series we have

$$\sum_{n=1}^{\infty} \frac{1}{(1+x)^n} = \frac{1}{1+x} + \frac{1}{(1+x)^2} + \frac{1}{(1+x)^3} + \cdots$$

This is a GS with $a = \frac{1}{1+x}$ & $r = \frac{1}{1+x}$. We are given

$$|r| = \left|\frac{1}{1+x}\right| = \frac{1}{|1+x|} < 1 \quad \text{because } x > 0$$

Applying $S_\infty = \frac{a}{1-r}$ we have

$$\sum_{n=1}^{\infty} \frac{1}{(1+x)^n} = \frac{1/1+x}{1-\frac{1}{1+x}}$$

$$= \frac{1}{1+x-1} = \frac{1}{x}$$

(multiplying numerator & denominator by $1+x$)

(d) Similarly we have

$$\sum_{n=1}^{\infty} \frac{1}{(1+x^2)^n} = \sum_{n=1}^{\infty} \left(\frac{1}{1+x^2}\right)^n$$

$$= \frac{1}{1+x^2} + \left(\frac{1}{1+x^2}\right)^2 + \left(\frac{1}{1+x^2}\right)^3 + \cdots$$

GS with $a = \frac{1}{1+x^2}$ & $r = \frac{1}{1+x^2}$

$|r| = \left|\frac{1}{1+x^2}\right| = \frac{1}{|1+x^2|} < 1$ because $1+x^2 > 1$ provided $x \neq 0$

Using the infinite sum formula we have

$$\sum_{n=1}^{\infty} \frac{1}{(1+x^2)^n} = \frac{1/1+x^2}{1-\frac{1}{1+x^2}}$$

$$= \frac{1}{1+x^2-1} = \frac{1}{x^2}$$