Chapter 3: Sequences of Functions

Convergence of Functions

By the end of this section you will be able to
- understand what is meant by pointwise convergence
- prove sequence of functions are pointwise convergent

So far we have examined sequences of real numbers. In this chapter we look at sequences of functions.
An example of a sequence of functions is

$$f_n(x)=\frac{x}{n}$$

Some of the terms of this sequence are:

$$f_1(x)=\frac{x}{1}=x\,,\ f_2(x)=\frac{x}{2},\ \ f_3(x)=\frac{x}{3},\ \ \ldots,\ \ f_{10}(x)=\frac{x}{10},\ \ \ldots,\ \ f_n(x)=\frac{x}{n},\ \ldots$$

These functions are straight lines of the form $y=mx+c$ with gradient $=\frac{1}{n}$ and vertical intercept equal to zero:

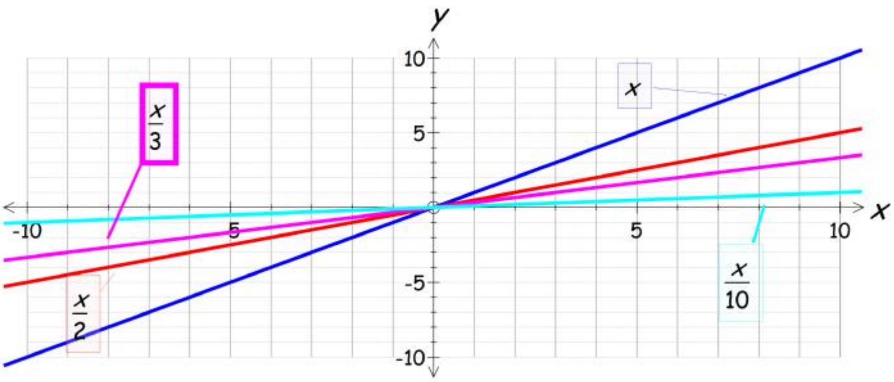


Fig 1
What do we mean by a sequence of functions?
The sequence $(f_1,\ f_2,\ f_3,\ \cdots,\ f_n,\ \cdots)$, or just (f_n), is a collection of functions that has the same domain, D say, and a range in $\mathbb{R}$. We write this as:

Definition (3.1). If for every $n\in\mathbb{N}$ there is a function $f_n:\ D\to\mathbb{R}$ where the domain D is a subset of $\mathbb{R}$ then $(f_1,\ f_2,\ f_3,\ \cdots,\ f_n,\ \cdots)$ is a **sequence of functions** on D to $\mathbb{R}$.

In the above example $f_n(x)=\frac{x}{n}$ the domain is $\mathbb{R}$ which means $f_n:\ \mathbb{R}\to\mathbb{R}$.

Example 1
Let $f_n:\ \mathbb{R}\to\mathbb{R}$ be a sequence of functions given by:

$$f_n(x)=1-\frac{x}{n}$$

Determine $f_n(x)$ for $n=10,\ \ 100,\ \ 1000$ and 5000.
Solution
We need to substitute $n=10,\ \ 100,\ \ 1000$ and 5000 into $f_n(x)=1-\frac{x}{n}$:

$$f_{10}(x)=1-\frac{x}{10},\quad f_{100}(x)=1-\frac{x}{100},\quad f_{1000}(x)=1-\frac{x}{1000} \text{ and } f_{5000}(x)=1-\frac{x}{5000}$$

We can plot these functions noting that they conform to the general straight line $y=mx+c$ where $y_n=f_n(x)=1-\frac{x}{n}$ has a vertical intercept of 1 and gradient of $-\frac{1}{n}$:

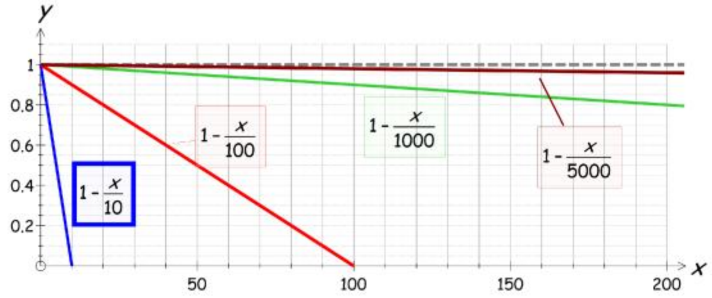


Fig 2

Does this sequence of functions $f_n(x)=1-\frac{x}{n}$ *converge as we increase the value of n?*

By examining the above graphs we note that as $n\to\infty$ the functions $f_n(x)=1-\frac{x}{n}$ seem to converge to $f(x)=1$. We **cannot** really claim this until we have defined **convergence** for functions.

A1 Convergence of Functions

In the remainder of this chapter we look at two types of convergence;

1) Pointwise convergence
2) Uniform convergence

In this section we examine pointwise convergence. Before we define what is meant by pointwise convergence we look at some examples of this.

Example 2

Let $f_n:[-1,\ 1]\to\mathbb{R}$ be a sequence of functions given by:

$$f_n(x)=x^n$$

(i) Plot $f_1(x)$, $f_2(x)$, $f_3(x)$ and $f_{10}(x)$ on the same axes.

(ii) What is the value of $f_n(x)=x^n$ as $n\to\infty$?

<u>Solution</u>

(i) Substituting $n=1,\ 2,\ 3$ and 10 into $f_n(x)=x^n$ we have

$$f_1(x)=x \text{ (linear)},\ f_2(x)=x^2 \text{ (quadratic)},\ f_3(x)=x^3 \text{ (cubic) and } f_{10}(x)=x^{10}$$

The graphs of these functions $f_1(x)$, $f_2(x)$, $f_3(x)$ and $f_{10}(x)$ are:

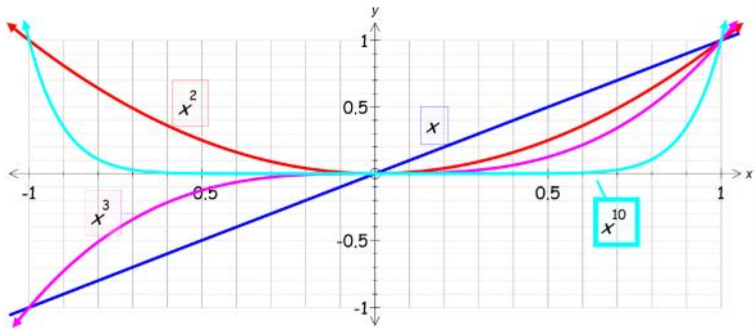


Fig 3

(ii) Our given domain is $x \in [-1, 1]$. *What does this mean?*

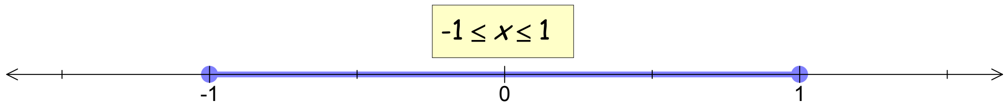


Fig 4

From our discussion on numerical sequences of chapter 2 we know that if $-1 < x < 1$ then $\lim_{n\to\infty}(x^n) = 0$. In other words, x^n converges to zero for all x such that $-1 < x < 1$ as $n \to \infty$.

However what is $f_n(x) = x^n$ equal to when $x = 1$?

We have $f_n(1) = 1^n = 1$. Hence $\lim_{n\to\infty}(1^n) = 1$.

What is $f_n(x) = x^n$ equal to when $x = -1$?

$$f_n(-1) = (-1)^n = (-1)^1, (-1)^2, (-1)^3, (-1)^4, \cdots$$
$$= -1, 1, -1, 1, \cdots$$

The sequence $f_n(x) = x^n$ oscillates between -1 and 1 which means it does **not** converge at $x = -1$.

Summarizing the above results we have:

$$\lim_{n\to\infty}[f_n(x)] = \lim_{n\to\infty}(x^n) = \begin{cases} 0 \ (\text{converges}) & \text{if } -1 < x < 1 \\ 1 \ (\text{converges}) & \text{if } x = 1 \\ \text{diverges} & \text{if } x = -1 \end{cases}$$

We can illustrate $\lim_{n\to\infty}[f_n(x)]$ as:

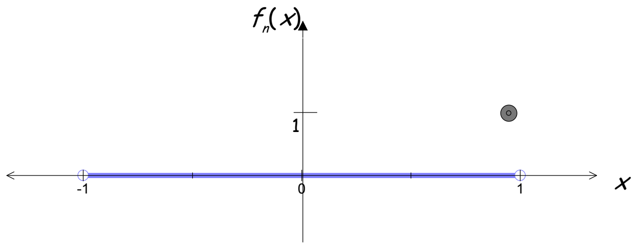


Fig 5

This means the sequence of functions shown in Fig 3 on page 2 converges to the function in Fig 5 above as *n* goes to infinity.

The given sequence of functions is said to (pointwise) converge for values of x: $-1 < x \le 1$.

Convergence or divergence of a sequence of functions $f_n(x)$ depends on the value of x as can be seen in the above example. For certain values of x in the domain, the sequence $f_n(x)$ may converge and for other values $f_n(x)$ may diverge as $n \to \infty$.

In the above example $f_n(x) = x^n$ converges to 0 for $-1 < x < 1$ and converges to 1 at $x = 1$.

The sequence diverges at $x = -1$.

We can write the results of the above example formally as:

$$\lim_{n\to\infty}(f_n(x)) = f(x) = \begin{cases} 0 & \text{if } -1 < x < 1 \\ 1 & \text{if } x = 1 \\ \text{diverges} & \text{if } x = -1 \end{cases}$$

For the above example we say $f_n(x)$ converges (pointwise) to $f(x)$ on $]-1, 1]$.

To say that the sequence of functions $f_n: D \to \mathbb{R}$ converges, means that for all x in D the sequence eventually converges to a real number which is denoted by $\lim_{n\to\infty}\left(f_n(x)\right)$.

We consider another example.

The sequence of functions given by $f_n(x) = \dfrac{x^{2n}}{1+x^{2n}}$ are illustrated below:

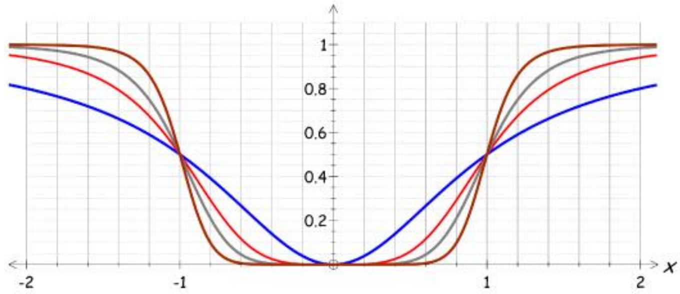
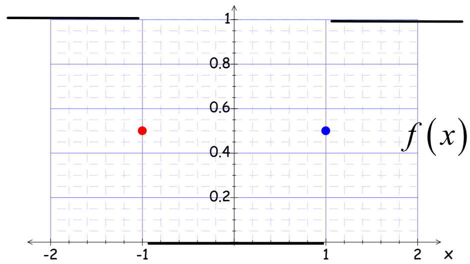


Fig 6

This sequence of functions converges to the following function $f(x)$:



$$f(x) = \lim_{n\to\infty}\left(f_n(x)\right) = \begin{cases} 0 & \text{if } -1 < x < 1 \\ 0.5 & \text{if } x = \pm 1 \\ 1 & \text{if } x > 1 \text{ and } x < -1 \end{cases}$$

Fig 7

A2 Calculations of Distance from the Limiting Function

We need to find the distance between the sequence of functions $\left(f_n(x)\right)$ and the limiting function $f(x)$ because if this distance can be made as small as we please for large enough n then we can say $f_n(x)$ converges to $f(x)$. The distance between $f_n(x)$ and $f(x)$ is determined by the value of n for a fixed value of x.

Example 3

Let $f_n: \mathbb{R} \to \mathbb{R}$ be a sequence of functions given by:

$$f_n(x) = 2 - \frac{x}{n}$$

(Intuitively, we can see that when x is fixed and as $n \to \infty$ the sequence of functions $f_n(x) = 2 - \dfrac{x}{n}$ tends to a limit of 2. In this example we find how close values of $f_n(x)$ are to the limiting value of 2 at $x = 1$. Note that if we fix a value of $x = 1$ say, then $f_n(1) = 2 - \dfrac{1}{n}$ is a numerical sequence.)

Find the following distances at $x = 1$:

(i) $\varepsilon_1 = \left|f_{10}(x) - 2\right|$ (ii) $\varepsilon_2 = \left|f_{100}(x) - 2\right|$ (iii) $\varepsilon_3 = \left|f_{1000}(x) - 2\right|$

<u>Solution</u>

In each case we substitute $x = 1$ into the given sequence.

i) For $n = 10$ we have

$$\varepsilon_1 = \left|f_{10}(1) - 2\right| = \left|\left(2 - \frac{1}{10}\right) - 2\right| = \left|-\frac{1}{10}\right| = \frac{1}{10} = 0.1$$

This means $f_{10}(1)$ is a distance of 0.1 from the limiting value of 2.

(ii) For $n = 100$ we have

$$\varepsilon_2 = \left|f_{100}(1) - 2\right| = \left|\left(2 - \frac{1}{100}\right) - 2\right| = \left|-\frac{1}{100}\right| = \frac{1}{100} = 0.01$$

(iii) For $n = 1000$ we have

$$\varepsilon_3 = \left|f_{1000}(1) - 2\right| = \left|\left(2 - \frac{1}{1000}\right) - 2\right| = \left|-\frac{1}{1000}\right| = \frac{1}{1000} = 0.001$$

What do these results mean?

(i) At $x = 1$ with $\varepsilon_1 = 0.1$ and $n = 10$, the function $f_n(x) = 2 - \frac{x}{n}$ lies between 1.9 and 2.1. This means that for $n > 10$ the sequence of functions $f_n(x)$ (this is $f_{11}(x),\ f_{12}(x), \cdots$) satisfy the inequality:

$$1.9 < f_n(x) = 1 - \frac{x}{n} < 2.1$$

We can represent this graphically for $\varepsilon_1 = 0.1$ as:

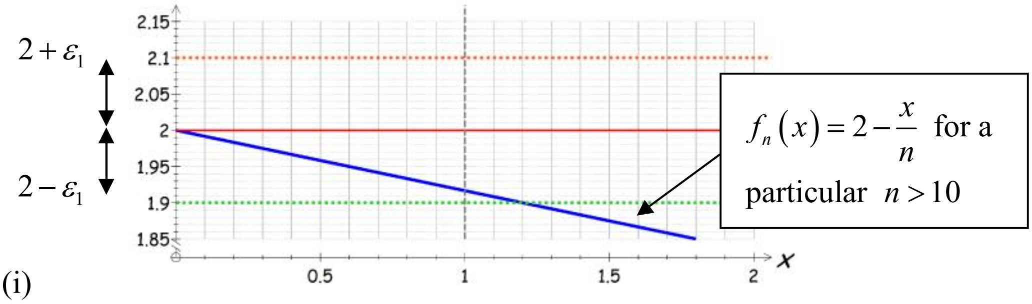


Fig 8 (i)

(ii) At $x = 1$ and $n = 100$, $f_n(x) = 2 - \frac{x}{n}$ is $\varepsilon_2 = 0.01$ away from its limit of 2. This means that for all $n > 100$, the sequence of functions $f_n(x) = 2 - \frac{x}{n}$ are within a distance of -0.01 and 0.01 of the limit of 2. Hence $1.99 < f_n(x) < 2.01$ for $n > 100$ and $x = 1$.

We can represent this graphically for $\varepsilon_2 = 0.01$ as:

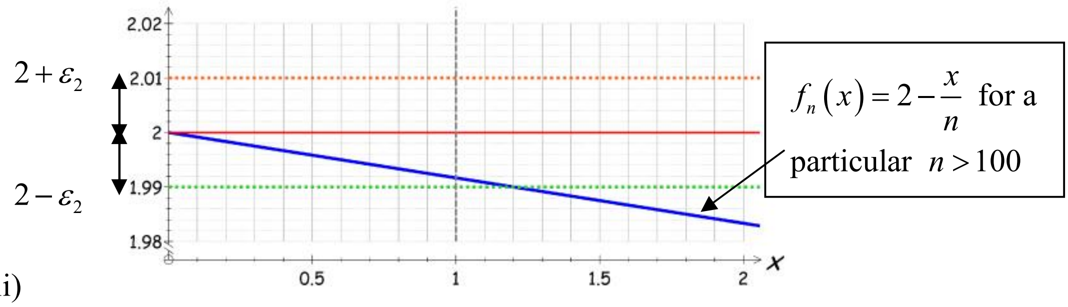


Fig 8 (ii)

(iii) At $x = 1$ and $n = 1000$, $f_n(x) = 2 - \frac{x}{n}$ is $\varepsilon_3 = 0.001$ away from its limit of 2. Hence for all $n > 1000$, the sequence of functions $f_n(x) = 2 - \frac{x}{n}$ are within a distance of -0.001 and 0.001 of the limit of 2. This means that for all $n > 1000$ the sequence of functions $f_n(x) = 2 - \frac{x}{n}$ at $x = 1$ lie between -0.001 and 0.001 of 2, that is we have

$$2-0.001=1.999<f_n(x)=2-\frac{x}{n}<2+0.001=2.001$$

The result for part (iii) where $\varepsilon_3=0.001$ is illustrated below:

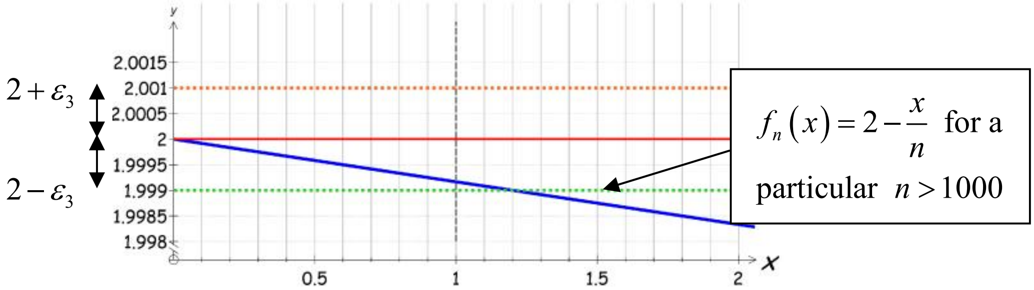


Fig 8 (iii)

Since in this example we had fixed *x* (=1) so the magnitude of *n* in the function $f_n(x)$ is only dependent on the given ε. For a smaller value of ε we need a larger value of *n*.

For pointwise convergence our *n* value depends on **both** the *x* value and the given $\varepsilon>0$.

You can observe in the above graphs (Fig 8 (i), (ii) and (iii)) at $x=1.5$ the sequence of functions $(f_n(x))$ do **not** lie within the given ε of the limiting function $f(x)=2$. You will need a larger value of *n* to ensure that at $x=1.5$ the sequence of functions $(f_n(x))$ lie within the given ε of 2.

A3 Definition of Pointwise Convergence of Functions

The formal definition of pointwise convergence is:

Definition (3.2).

The sequence $(f_n(x))$ converges **pointwise** to a function $f(x)$ in the domain *D* $\Leftrightarrow$ for every x in *D* and for every $\varepsilon>0$ there exists a natural number N_0 (depending on *x* and ε) such that

$$|f_n(x)-f(x)|<\varepsilon \text{ provided } n\geq N_0$$

In this definition the natural number N_0 depends on our choice of *x* and ε. Also, when $(f_n(x))$ converges **pointwise** to a function $f(x)$ it is normally denoted by

$$\lim_{n\to\infty}(f_n(x))=f(x)$$

The graph below illustrates this definition. For $n\geq N_0$ the functions $f_n(x)$ are within ε away from the limiting function $f(x)$. This means the functions will eventually be at most a distance of ε away from $f(x)$. The graphs below shows the sequence of functions $(f_n(x))$ at $x=c$ converging pointwise to $f(x)$:

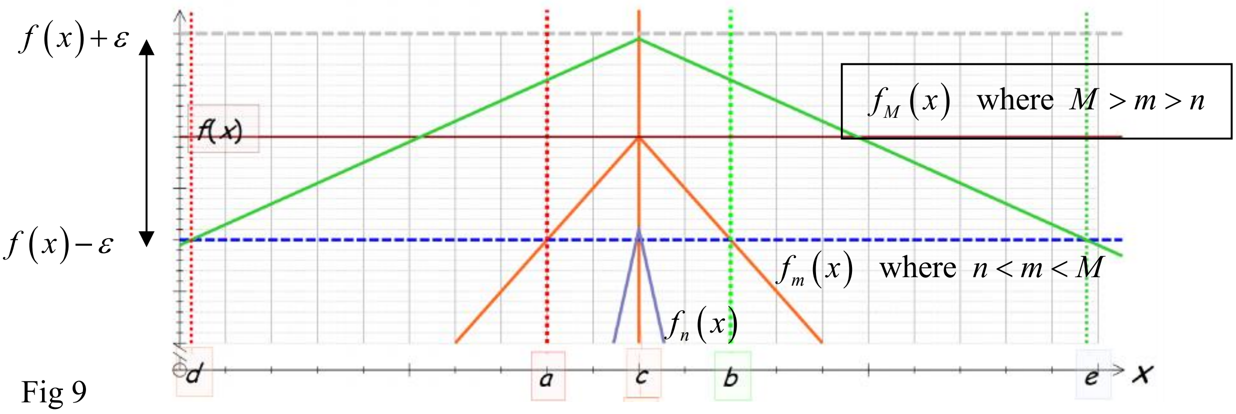


Fig 9

Fig 9 shows that $f_n(x)$ for $n \geq N_0$ converges to $f(x)$ at the point $x = c$. For $m > n$ the functions $f_m(x)$ converges to $f(x)$ within the interval $x \in \left]a,\ b\right[$ because all the functions $f_m(x)$ for $m > n \geq N_0$ lie within $\varepsilon > 0$ of $f(x)$.

Note that the sequence of functions $f_M(x)$ for $M > m > n \geq N_0$ converge to $f(x)$ for a larger interval $x \in \left]d,\ e\right[$.

Remember **pointwise** convergence **depends** on the value of x as well as the value of $\varepsilon > 0$.

A4 Applying the Defintion

We use the above definition to show that a particular sequence of functions converges pointwise.

Example 4

Let $f_n : \mathbb{R} \to \mathbb{R}$ be a sequence of functions given by:

$$f_n(x) = \frac{x}{n}$$

By using Definition (3.2) show that this sequence of functions converges to the zero function on $\mathbb{R}$.

Solution

We need to show that $f_n(x) = \dfrac{x}{n}$ converges to $f(x) = 0$ for **all** x in $\mathbb{R}$.

Let $\varepsilon > 0$ be given. Then there is a natural number N_0 such that for all $n \geq N_0$ we have

$$\left|f_n(x) - f(x)\right| = \left|\frac{x}{n} - 0\right| = \frac{|x|}{n} \leq \frac{|x|}{N_0}$$

To show that $f_n(x) = \dfrac{x}{n}$ converges to zero we need to find a N_0 which gives $\dfrac{|x|}{N_0} < \varepsilon$.

Transposing this gives the inequality $\dfrac{|x|}{\varepsilon} < N_0$ or writing this the other way $N_0 > \dfrac{|x|}{\varepsilon}$. Hence for all $n \geq N_0$ we have

$$\left|f_n(x) - f(x)\right| \leq \frac{|x|}{N_0} < \varepsilon \quad \text{provided } N_0 > \frac{|x|}{\varepsilon}$$

By Definition (3.2) on page 6:

The sequence $(f_n(x))$ converges **pointwise** to a function $f(x)$ on the domain $D \Leftrightarrow$ for every x in D and for every $\varepsilon > 0$ there exists a natural number N_0 such that

$$\left|f_n(x) - f(x)\right| < \varepsilon \quad \text{provided } n \geq N_0$$

We conclude that sequence of functions $f_n(x) = \dfrac{x}{n}$ converges pointwise to the zero function, $f(x) = 0$, for all x in $\mathbb{R}$.

Here we examine some numerical values for the above example. *For* $x = 2$ *and* $\varepsilon = 0.0001$ *what natural number* N_0 *ensures that* $f_n(x) = \dfrac{x}{n}$ *is within* $\varepsilon = 0.0001$ *of the limit zero?*

We need to find $n \geq N_0$ such that at $x = 2$ we have

$$-0.0001 < f_n(x) = \frac{2}{n} < 0.0001$$

By the answer given in the above example we select our $N_0 > \frac{|x|}{\varepsilon} = \frac{2}{0.0001} = 20\,000$. We need a $N_0 > 20\,000$ so if we chose $N_0 = 20\ 001$ then

$$-0.0001 < f_{20\,001}(x) = \frac{2}{20\ 001} < 0.0001$$

What value of N_0 can we chose for $x = 3$ with the same $\varepsilon = 0.0001$?

$$N_0 > \frac{|x|}{\varepsilon} = \frac{3}{0.0001} = 30\,000$$

For $N_0 = 30\ 001$ and $n \geq N_0 = 30\ 001$ we have $-0.0001 < f_{30\,001}(x) = \dfrac{3}{30\ 001} < 0.0001$.

We can illustrate $f_n(x)$ for $\varepsilon = 0.0001$ by plotting graphs for $n = 21\,000$ and $31\ 000$:

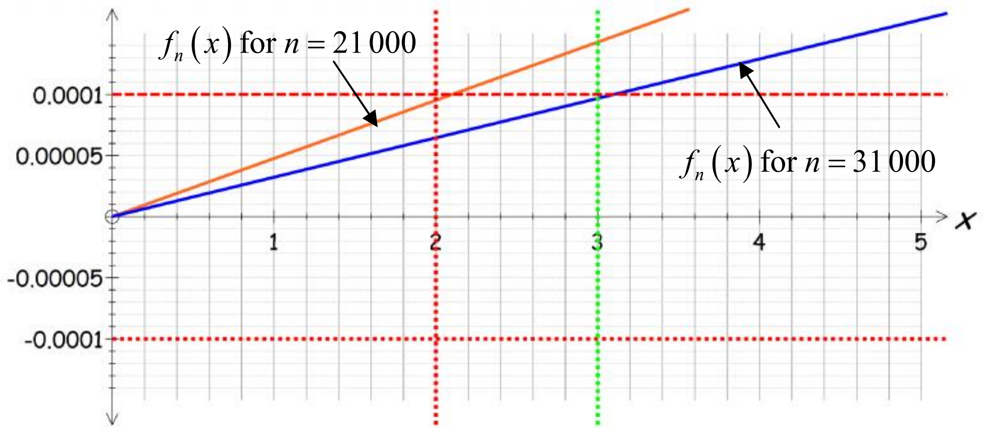


Fig 10

In this example we have our $\varepsilon = 0.0001$ fixed so the $n \geq N_0$ depends on the value of *x*. For larger values of *x* we need a greater N_0 to be within $\varepsilon = 0.0001$ of the limit function zero. For $x = 2$ we need $N_0 > 20\,000$ and for $x = 3$ we need $N_0 > 30\,000$.

What value does N_0 need to be greater than at $x = 10$ to be within the same $\varepsilon = 0.0001$ of zero?

From the above example $N_0 > \dfrac{|x|}{\varepsilon}$ so we need $N_0 > \dfrac{|x|}{\varepsilon} = \dfrac{10}{0.0001} = 100\,000$. For different values of *x* we require different values of N_0.

Example 5

Let $f_n : \mathbb{R} \to \mathbb{R}$ be a sequence of functions given by:

$$f_n(x) = 1 - \frac{x}{n}$$

By using Definition (3.2) show that this sequence of functions converges to 1 on $\mathbb{R}$.

<u>Solution</u>

How do we show that $f_n(x) = 1 - \dfrac{x}{n}$ converge to $f(x) = 1$ for all x in $\mathbb{R}$?

Let $\varepsilon > 0$ be given. Then there is a natural number N_0 such that for all $n \geq N_0$:

$$\left|f_n(x)-1\right| = \left|1-\frac{x}{n}-1\right| = \left|-\frac{x}{n}\right| = \left|\frac{x}{n}\right| \le \frac{|x|}{N_0} \qquad (\dagger)$$

We need to find the natural number N_0 such that $\frac{|x|}{N_0} < \varepsilon$. Transposing gives

$$\frac{|x|}{N_0} < \varepsilon \quad \Leftrightarrow \quad \frac{|x|}{\varepsilon} < N_0 \text{ or } N_0 > \frac{|x|}{\varepsilon}$$

For $n \ge N_0 > \frac{|x|}{\varepsilon}$ we have

$$\left|f_n(x)-1\right| \underbrace{\le}_{\text{By } (\dagger)} \frac{|x|}{N_0} < \varepsilon \quad \left[\text{Because } N_0 > \frac{|x|}{\varepsilon}\right]$$

By Definition (3.2) on page 6 the given sequence of functions $\left(f_n(x)\right)$ converges pointwise to the function $f(x)=1$ on $\mathbb{R}$.

Example 6

Let $f_n : \mathbb{R} \to \mathbb{R}$ be a sequence of functions given by:

$$f_n(x) = \frac{\sin(nx)}{n}$$

By using Definition (3.2) show that this sequence of functions converges to the zero function $f(x)=0$ on $\mathbb{R}$.

Solution

Let $\varepsilon > 0$ be given and N_0 be a natural number such that for all $n \ge N_0$ we have

$$\left|f_n(x)-0\right| = \left|\frac{\sin(nx)}{n}-0\right| = \left|\frac{\sin(nx)}{n}\right| = \frac{\left|\sin(nx)\right|}{n} \qquad (*)$$

How can we show this is less than ε*?*

From the properties of sine function we know that $\left|\sin(nx)\right| \le 1$. Substituting this inequality into (*) and using $n \ge N_0$ we have

$$\left|f_n(x)-0\right| = \frac{\left|\sin(nx)\right|}{n} \le \frac{1}{n} \le \frac{1}{N_0}$$

We need to show that $\frac{1}{N_0} < \varepsilon$. Transposing this gives

$$\frac{1}{N_0} < \varepsilon \quad \Leftrightarrow \quad \frac{1}{\varepsilon} < N_0 \quad \text{or} \quad N_0 > \frac{1}{\varepsilon}$$

Hence for $n \ge N_0 > \frac{1}{\varepsilon}$ we have

$$\left|f_n(x)-0\right| = \frac{\left|\sin(nx)\right|}{n} \le \frac{1}{N_0} < \varepsilon$$

By definition (3.2) the sequence of functions $f_n(x)=\dfrac{\sin(nx)}{n}$ converges to $f(x)=0$ for all x in $\mathbb{R}$. Note that in this case the natural number N_0 does **not** depend on the value of x. This is actually an example of uniform convergence which is discussed in the next section.

We can apply the results of numerical sequences to show that given sequences of functions are convergent. For example we know from our work on numerical sequences that

$$\lim_{n\to\infty}\left(\frac{1}{n}\right)=\lim_{n\to\infty}\left(\frac{1}{n^2}\right)=0$$

We use these results in the next example. Using these results often makes task of proving convergence a lot easier.

Example 7

Let $f_n : \mathbb{R}\to\mathbb{R}$ be a sequence of functions given by:

$$f_n(x)=\frac{(x+n)^2}{n^2}$$

Show that this sequence converges to 0 as $n\to\infty$ for all x in $\mathbb{R}$.

Solution

We can apply the limits of sequences to show this result. We have

$$\begin{aligned}\lim_{n\to\infty}\left[f_n(x)\right]&=\lim_{n\to\infty}\left[\frac{(x+n)^2}{n^2}\right]\\&=\lim_{n\to\infty}\left[\frac{x^2+2nx+n^2}{n^2}\right]\\&=\lim_{n\to\infty}\left[\frac{x^2}{n^2}+\frac{2nx}{n^2}+\frac{n^2}{n^2}\right]\\&=x^2\left(\lim_{n\to\infty}\left(\frac{1}{n^2}\right)\right)+2x\left(\lim_{n\to\infty}\left(\frac{1}{n}\right)\right)+1\\&=x^2(0)+x(0)+1=1\quad\left[\text{Because }\lim_{n\to\infty}\left(\frac{1}{n^2}\right)=\lim_{n\to\infty}\left(\frac{1}{n}\right)=0\right]\end{aligned}$$

Example 8

Let $f_n : [0,\ \pi]\to\mathbb{R}$ be a sequence of functions given by:

$$f_n(x)=\sin^n(x)$$

Determine the pointwise convergence of these functions.

Solution

By the properties of the sine function we know that $|\sin(x)|\le 1$ for $x\in[0,\ \pi]$. Also $\sin^n(x)=[\sin(x)]^n$. At $x=\dfrac{\pi}{2}$ we have $\sin\left(\dfrac{\pi}{2}\right)=1$ and $1^n=1$. For the rest of the values of x, that is $0\le x<\dfrac{\pi}{2}$ and $\dfrac{\pi}{2}<x\le\pi$ (all values of x between 0 and π apart from $\pi/2$) we have

$$\lim_{n\to\infty}\left([\sin(x)]^n\right)=0 \text{ because } |\sin(x)|\le 1$$

The functions $f_n(x) = \sin^n(x)$ converge pointwise to 0 if $0 \le x < \frac{\pi}{2}$ and $\frac{\pi}{2} < x \le \pi$ and 1 at $x = \frac{\pi}{2}$. Hence we can write this as $\lim_{n\to\infty}\left[f_n(x)\right] = \lim_{n\to\infty}\left[\sin^n(x)\right] = f(x)$ where

$$f(x) = \begin{cases} 0 & \text{if } 0 \le x < \frac{\pi}{2} \text{ and } \frac{\pi}{2} < x \le \pi \\ 1 & \text{if } x = \pi/2 \end{cases}$$

SUMMARY

The sequence of functions $f_n(x)$ converges **pointwise** to $f(x)$ means that for all $n \ge N_0$

$$\left|f_n(x) - f(x)\right| < \varepsilon$$

where the N_0 depends on *x* and a given $\varepsilon > 0$.