SECTION G **Principle of Mathematical Induction**

By the end of this section you will be able to

- understand the procedure for proof by induction
- construct proofs by induction
- prove equality of various sums of natural numbers

In this section we examine propositions concerning positive integers. The positive integers 1, 2, 3, 4,…are called **natural numbers** or **counting numbers**. In this section lower case letters represent natural numbers.

G1 Principle of Mathematical Induction

Mathematical induction is a powerful tool used to prove propositions concerning natural numbers.

Principle of Mathematical Induction (1.18)

For each natural number *n*, let $P(n)$ be a proposition about *n*. If $P(n)$ satisfies:

1) $P(1)$ is true
2) For an arbitrary k, $P(k)$ is true implies $P(k+1)$ is true

Then for **all** natural numbers, n, we have $P(n)$ is true.

Parts 1) and 2) suggest that

$P(1)$ implies $P(2)$, $P(2)$ implies $P(3)$, $P(3)$ implies $P(4)$, $P(4)$ implies $P(5)$,...

This is sometimes called the domino effect. Once one of the dominos topples it causes the rest to topple as well.

Fig 6 Domino Effect

The process is that we show $P(1)$ is true and by assuming $P(k)$ is true we prove $P(k+1)$ is true. If **both** $P(1)$ is true and $P(k)$ implies $P(k+1)$ then proposition $P(n)$ is true for all natural numbers *n*.

We can apply this principle of mathematical induction to prove results about natural numbers.

G2 Examples

Example 43

For every natural number *n* prove the proposition $P(n)$ given by

$$1+2+3+4+...+n=\frac{1}{2}n(n+1)$$

Comment

What does this proposition mean?

It means that if we add the first n natural numbers then the answer will be $\frac{1}{2}n(n+1)$.

For example if we add the first 2 numbers we have

$$1+2=\frac{1}{2}(2)(2+1)=3 \qquad \text{[Substituting } n=2 \text{ into the above]}$$

Similarly we have

$$\underbrace{1+2+3}_{\text{3 Terms}}=\frac{1}{2}(3)(3+1)=6 \qquad \left[\begin{array}{l}\text{Adding the first 3 natural numbers,}\\ \text{that is } n=3\end{array}\right]$$

$$\underbrace{1+2+3+4}_{\text{4 Terms}}=\frac{1}{2}(4)(4+1)=10 \qquad \left[\begin{array}{l}\text{Adding the first 4 natural numbers,}\\ \text{that is } n=4\end{array}\right]$$

$$\underbrace{1+2+3+4+5}_{\text{5 Terms}}=\frac{1}{2}(5)(5+1)=15 \qquad \left[\begin{array}{l}\text{Adding the first 5 natural numbers,}\\ \text{that is } n=5\end{array}\right]$$

and so on. We need to show this result for all the natural numbers n. *How?*

Employ mathematical induction because the proposition concerns the natural numbers n.

Proof

First we check the proposition for $n=1$:

$$1=\frac{1}{2}(1)(1+1) \qquad \surd$$

Hence the proposition is true for $n=1$. Next we **assume** the given proposition is true for $n=k$, that is $P(k)$. *How do we write this* $P(k)$*?*

By substituting $n=k$ into the given proposition

$$1+2+3+4+...+n=\frac{1}{2}n(n+1)$$

which gives

$$1+2+3+4+...+k=\frac{1}{2}(k)(k+1) \qquad (*)$$

We have labelled this result by (*) because we are going to prove the proposition for $n=k+1$ by using (*). *How do we write the proposition* $P(k+1)$*?*

By substituting $n=k+1$ into the given proposition, $1+2+3+4+...+n=\frac{1}{2}n(n+1)$:

$$1+2+3+4+...+k+(k+1)=\frac{1}{2}(k+1)\underbrace{(k+1+1)}_{=k+2}=\frac{1}{2}(k+1)(k+2) \qquad (**)$$

This means that **we have to prove** the sum of the first $(k+1)$ natural numbers is equal to $\frac{1}{2}(k+1)(k+2)$. It is critical that you realise we need to prove (**). We have **only** stated $P(k+1)$ **not** proven it yet. The challenge is to show that the Left Hand Side is equal to the Right Hand Side of (**). *How?*
We can simplify the sum up to the first k terms by using (*), hence we have

$$1+2+3+4+...+k+(k+1) = \underbrace{1+2+3+4+...+k}_{=\frac{1}{2}k(k+1) \text{ by } (*)} + (k+1)$$

$$= \frac{1}{2}k(k+1)+(k+1) \qquad \text{[Simplifying]}$$

$$= \frac{1}{2}(k+1)k+\frac{1}{2}(k+1)2 \qquad \left[\text{Rewriting } (k+1)=\frac{1}{2}(k+1)2\right]$$

$$= \frac{1}{2}(k+1)(k+2) \qquad \text{[Factorizing]}$$

The last line is the Right Hand Side of (**). Hence our result holds by the principle of mathematical induction (1.18) because we have shown (**).

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Notice how we assume $P(k)$ to be true and then use it to prove $P(k+1)$. The proposition $P(k)$ in the above was (*) and we used this in the derivation of $P(k+1)$ which was

$$1+2+3+4+...+k+(k+1)=\frac{1}{2}(k+1)(k+2) \qquad (**)$$

Since $P(1)$ is true and $P(k)$ implies $P(k+1)$ is true therefore we have the required result, $1+2+3+4+...+n=\frac{1}{2}n(n+1)$, by mathematical induction.

Example 44

For every natural number n prove the proposition $P(n)$ given by

$$1+3+5+7+...+(2n-1)=n^2$$

Comment. *What does this proposition mean?*
It means that if we add the first n odd counting numbers then the answer will be the square of n. For example if we add the first 2 odd counting numbers we have

$$1+3=2^2=4 \qquad \text{[Substituting } n=2 \text{ into the given proposition]}$$

Similarly we have

$$\underbrace{1+3+5}_{3 \text{ Terms}}=3^2 \qquad \text{[Adding the first 3 odd numbers, that is } n=3\text{]}$$

$$\underbrace{1+3+5+7}_{4 \text{ Terms}}=4^2 \qquad \text{[Adding the first 4 odd numbers, that is } n=4\text{]}$$

$$\underbrace{1+3+5+7+9}_{5 \text{ Terms}}=5^2 \qquad \text{[Adding the first 5 odd numbers, that is } n=5\text{]}$$

and so on. We need to show this result for all the natural numbers n. *How?*
Apply mathematical induction because the proposition is valid for all the natural numbers.

Proof

First we check the proposition for $n=1$:

$$1 = 1^2 \quad \surd$$

Hence the proposition is true for $P(1)$. Next we **assume** the given proposition is true for $n=k$ that is $P(k)$:

$$\underbrace{1+3+5+7+\ldots+(2k-1)}_{k \text{ terms}} = k^2 \qquad (\dagger)$$

We have labelled $P(k)$ by (†) because we are going to prove the proposition for $n=k+1$ by using (†). *How do we write the proposition* $P(k+1)$*?*

By substituting $n=k+1$ into the given proposition, $1+3+5+7+\ldots+(2n-1)=n^2$:

$$\underbrace{1+3+5+7+\ldots+(2k-1)}_{\text{First } k \text{ terms}} + \underbrace{(2(k+1)-1)}_{(k+1)\text{th term}} = (k+1)^2 \qquad (\dagger\dagger)$$

We need to prove this, (††), result. The challenge is to show that the Left Hand Side is equal to the Right Hand Side of (††). *How?*

We can simplify by writing the sum up to $(2k-1)$ by using (†), hence we have

$$\underbrace{1+3+5+7+\ldots+(2k-1)}_{\text{First } k \text{ terms}} + (2(k+1)-1) = \underbrace{1+3+5+7+\ldots+(2k-1)}_{=k^2 \text{ by } (\dagger)} + (2k+2-1)$$

$$= k^2+2k+1 \qquad [\text{Simplifying}]$$

$$= (k+1)^2 \qquad [\text{Factorizing}]$$

The last line is the same as the Right Hand Side of (††). By the principle of mathematical induction we have our required result because we have shown (††).

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In the above example we first showed that the given result was true for $P(1)$:

$$1 = 1^2 \qquad [n=1]$$

Secondly we assumed it is true for $P(k)$:

$$1+3+5+7+\ldots+(2k-1)=k^2 \qquad [n=k]$$

and finally we used this assumption to produce the result for $P(k+1)$:

$$1+3+5+7+\ldots+(2k-1)+(2(k+1)-1)=(k+1)^2 \qquad [n=k+1]$$

The given result, $1+3+5+7+\ldots+(2n-1)=n^2$, follows by the principle of mathematical induction.

Example 45

Prove that for every natural number n the proposition $P(n)$ given by

$$1^2+2^2+3^2+\ldots+n^2=\frac{n(n+1)(2n+1)}{6}$$

We use mathematical induction but it requires a lot more algebraic manipulation than the previous two examples. *How does mathematical induction work for this case?*

Proof

We first show the proposition is true for $n=1$ by substituting this into $P(n)$:

$$1^2=\frac{1(1+1)(2+1)}{6} \qquad \surd$$

Clearly $P(1)$ is correct. *What do we do next?*

Assume the given proposition is true for $n=k$, that is $P(k)$:

$$1^2+2^2+3^2+\ldots+k^2=\frac{k(k+1)(2k+1)}{6} \qquad (\clubsuit)$$

The 'meat' in mathematical induction is to show the given proposition is true for $n=k+1$ by employing (♣). *How?*

We first write down what **we need** to prove, that is write down the proposition $P(k+1)$ by substituting $n=k+1$ into the given proposition:

$$1^2+2^2+3^2+\ldots+k^2+(k+1)^2=\frac{(k+1)(k+1+1)(2(k+1)+1)}{6}$$

$$=\frac{(k+1)(k+2)(2k+3)}{6} \qquad (\clubsuit\clubsuit)$$

What do we need to show?

The Left Hand Side is equal to the Right Hand Side of (♣♣) by using (♣). *How?*

We know by (♣) that the sum up to k^2 is equal to $\frac{k(k+1)(2k+1)}{6}$, so we use this and then apply algebraic manipulation to get the Right Hand Side:

$$\underbrace{1^2+2^2+3^2+\ldots+k^2}_{=\frac{k(k+1)(2k+1)}{6}}+(k+1)^2=\frac{k(k+1)(2k+1)}{6}+(k+1)^2$$

$$=\frac{k(k+1)(2k+1)}{6}+\frac{6(k+1)^2}{6} \qquad \text{[Common Denominator]}$$

$$=\frac{k+1}{6}\left[k(2k+1)+6(k+1)\right] \qquad \left[\text{Factorizing out } \frac{k+1}{6}\right]$$

$$=\frac{k+1}{6}\left[2k^2+\underbrace{k+6k}_{=7k}+6\right] \qquad \left[\text{Expanding inside the Square Brackets}\right]$$

$$=\frac{k+1}{6}\left[2k^2+7k+6\right] \qquad \left[\text{Simplifying the Quadratic}\right]$$

$$=\frac{k+1}{6}(k+2)(2k+3) \qquad \left[\text{Factorizing the Quadratic}\right]$$

$$=\frac{(k+1)(k+2)(2k+3)}{6}$$

The last line is the same as the Right Hand Side of (♣♣). Hence we have proved our required result.

■

In the above example we first checked that the result was true for $n=1$. Then we assumed the result was true for $n=k$. Finally using this assumption we proved the result was true for $n=k+1$. By mathematical induction we have proven

$$1^2+2^2+3^2+\ldots+n^2=\frac{n(n+1)(2n+1)}{6}$$

SUMMARY

We use mathematical induction to prove propositions involving natural numbers.

The principle of mathematical induction to prove a proposition $P(n)$ involves

1. Showing the result for $n=1$, that is $P(1)$.
2. Assuming the result is true for $n=k$ where k is an arbitrary integer, that is assuming $P(k)$ is true.
3. Prove the result for $n=k+1$, that is prove $P(k+1)$. This normally requires us to use the assumption $P(k)$ in part 2.