Section F **Monotone Sequences**

By the end of this section you will be able to
- understand what is meant by the term monotone sequence
- prove properties involving bounded monotone sequences
- find limits of a bounded monotone sequence

For understanding this section you will need to know the procedure for proof by induction and rules of inequalities. We use these concepts throughout this section.

F1 **Definition of a Monotone Sequence**

The following are examples of an increasing sequence:

a) 1, 2, 3, 4, …, n, …
b) 1, 2, 2, 3, 3, 3, 4, 4, 5, 5, 5, …
c) 1, 1, 1, 1, 1, 1, 1,…
d) $x,\ x^2,\ x^3,\ x^4,\ x^5,...$ where $x \ge 1$

What do you think is the definition of an ***increasing sequence****?*

Definition (5.21). A sequence (x_n) is an **increasing sequence** if

$$x_1 \le x_2 \le x_3 \le x_4 \le x_5 \le x_6 \cdots \le x_n \le x_{n+1} \le \cdots$$

The **smallest** value of an increasing sequence is x_1 and as n gets larger so does the value of x_n. *Can you think of any other examples of increasing sequences?*

$x_n = 2, 4, 6, 8, 10, 12,\ldots$

$x_n = 2, 4, 8, 16, 32, 64,\ldots$

$x_n = 1, 1, 2, 3, 5, 8, 13, 21, 34, 55,\ldots$

$x_n = \frac{1}{81},\ \frac{1}{27},\ \frac{1}{9},\ \frac{1}{3},\ 1,\ 3,\cdots$

The following are examples of a decreasing sequence:

a) $-1,\ -2,\ -3,\ -4,\ \ldots, -n,\ \ldots$
b) $1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \frac{1}{5},\ \frac{1}{6},\cdots,\frac{1}{n},\ \frac{1}{n+1},\cdots$
c) 1, 1, 1, 1, 1, 1, 1,…
d) $x,\ x^2,\ x^3,\ x^4,\ x^5,...$ where $0 \le x \le 1$

What do you think is the definition of a ***decreasing sequence****?*

Definition (5.22). A sequence (x_n) is a **decreasing sequence** if

$$x_1 \ge x_2 \ge x_3 \ge x_4 \ge x_5 \ge x_6 \cdots \ge x_n \ge x_{n+1} \ge \cdots$$

The **largest** value of a decreasing sequence is x_1 and as n gets larger the value of x_n gets smaller. Other examples of decreasing sequences are:

$x_n = 23, 19, 17, 13, 11, 7, \ldots$

$x_n = 1,\ \frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ \frac{1}{16},\ \frac{1}{32},\cdots$

$x_n = 0,\ -1,\ -2,\ -3,\ -4,\ -5,\cdots$

$x_n = 1,\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{3},\ \frac{1}{3},\ \frac{1}{4},\ \frac{1}{4},\ \frac{1}{4},\ \frac{1}{4},\ \frac{1}{5},\ldots$

Increasing and **decreasing** sequences are examples of **monotone** sequences.

*What does the word **monotone** mean?*
Monotone is something that does **not** change – stays the same. Hence this is why an increasing sequence is a monotonic sequence because it keeps on increasing or remains constant. Similarly decreasing sequence does **not** change direction, it keeps on decreasing or remains constant, so it is a monotonic sequence.
Monotonic sequences have a particularly useful property which we discuss in the next section.

F2 **A Useful Property of a Bounded Monotone Sequence**

We prove what is called the "Monotone Convergence Theorem" which says that a bounded monotonic sequence is convergent. To prove this we use the following proposition from an **earlier** section:

Proposition (5.5). Let S be a non-empty subset of real numbers. A real number L is the supremum (Least Upper Bound) of the set S if and only if
(i) For all $x \in S$ we have $x \le L$
(ii) For every $\varepsilon > 0$ there is a real number y in the set S such that

$$L - \varepsilon < y \le L$$

If you don't understand this proposition then go over the proof of this in section 5B.

Monotone Convergence Theorem (5.23).
A **bounded monotone** sequence of real numbers is **convergent**.
Proof.
Without loss of generality assume (x_n) is a **bounded increasing** sequence. Therefore the set consisting of the sequence (x_n) is bounded above which means it has a supremum (Least Upper Bound) call it L. By the above proposition (5.5) part (ii) there is an element say $y = x_{N_0+1}$ of the given sequence such that

$$L - \varepsilon < x_{N_0+1}$$

for an arbitrary $\varepsilon > 0$. Since (x_n) is an **increasing** sequence we have for all $n > N_0$

$$L - \varepsilon < x_{N_0+1} \le x_n \qquad (*)$$

By the above proposition (5.5) part (i) we also have for all $n \in \mathbb{N}$

$$x_n \le L < L + \varepsilon \qquad (**)$$

By combining these inequalities, (*) and (**), we have for all $n > N_0$ the inequality

$$L - \varepsilon < x_n < L + \varepsilon$$

This means that $|x_n - L| < \varepsilon$ for all $n > N_0$ which implies the sequence (x_n) converges to the limit L. Hence a bounded monotone sequence converges.

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You are asked to prove the Monotone Convergence Theorem (5.23) for a **bounded decreasing** sequence in Exercise 5(f).

Example 24

Prove that the sequence $x_n = 1 - \frac{1}{n}$ is convergent by using the Monotone Convergence Theorem.

Solution.

We need to show that the given sequence $x_n = 1 - \frac{1}{n}$ is a bounded and monotonic sequence. We can evaluate the first few terms of the sequence $x_n = 1 - \frac{1}{n}$ by substituting $n = 1,\ 2,\ 3,\ 4, \cdots$

$$x_1 = 1 - \frac{1}{1} = 0 \qquad \left[\text{Substituting } n = 1 \text{ into } x_n = 1 - \frac{1}{n}\right]$$

$$x_2 = 1 - \frac{1}{2} = \frac{1}{2} \qquad \left[\text{Substituting } n = 2 \text{ into } x_n = 1 - \frac{1}{n}\right]$$

$$x_3 = 1 - \frac{1}{3} = \frac{2}{3} \qquad \left[\text{Substituting } n = 3 \text{ into } x_n = 1 - \frac{1}{n}\right]$$

$$x_4 = 1 - \frac{1}{4} = \frac{3}{4} \qquad \left[\text{Substituting } n = 4 \text{ into } x_n = 1 - \frac{1}{n}\right]$$

Looks as if the given sequence (x_n) has an upper bound of 1 which means that for all $n \in \mathbb{N}$

$$x_n < 1$$

How do we show this result?

We show that $x_n - 1 < 0$. Since $n \geq 1$ we have

$$x_n - 1 = \underbrace{1 - \frac{1}{n}}_{= x_n} - 1$$

$$= -\frac{1}{n} < 0 \qquad [\text{Because } n \geq 1]$$

Hence $x_n - 1 < 0$ which gives $x_n < 1$, we have our required result, that is for all $n \in \mathbb{N}$ $x_n < 1$. We conclude the given sequence (x_n) is bounded.

Need to show that the given sequence $x_n = 1 - \frac{1}{n}$ is increasing (monotonic). *How?*

Required to prove that $x_{n+1} \geq x_n$ for all $n \in \mathbb{N}$. We show that $x_{n+1} - x_n \geq 0$.

From the given sequence we have $x_{n+1} = 1 - \frac{1}{n+1}$ and $x_n = 1 - \frac{1}{n}$:

$$x_{n+1} - x_n = 1 - \frac{1}{n+1} - \left(1 - \frac{1}{n}\right)$$

$$= \frac{1}{n} - \frac{1}{n+1} \qquad [\text{Simplifying}]$$

$$> 0 \qquad \left[\text{Because } n < n+1 \iff \frac{1}{n} > \frac{1}{n+1}\right]$$

Hence we have $x_{n+1} - x_n > 0$ which means $x_{n+1} > x_n$ therefore (x_n) is an increasing sequence. We have shown that the given sequence $x_n = 1 - \frac{1}{n}$ is bounded and increasing which means it is a bounded monotonic sequence therefore by the Monotone Convergence Theorem (5.23) we conclude that the sequence $x_n = 1 - \frac{1}{n}$ converges.

Actually the sequence (x_n) in the above Example 24 converges to 1. That is

$\lim_{n\to\infty}\left(1-\frac{1}{n}\right)=1$.

A bounded monotone sequence (x_n) converges to the supremum (Least Upper Bound) of the set $\{x_n \mid n \in \mathbb{N}\}$. In Example 24 the supremum of the set $\left\{\left(1-\frac{1}{n}\right) \;\middle|\; n\in\mathbb{N}\right\}$ is 1 therefore the sequence $x_n = 1-\frac{1}{n}$ converges to 1 that is $\lim_{n\to\infty}\left(1-\frac{1}{n}\right)=1$.

We can state this and the infimum of a bounded decreasing sequence as general results.

F3 **Limits of a Bounded Monotone Sequence**

Proposition (5.24). If (x_n) is a bounded increasing sequence of real numbers then

$$\lim_{n\to\infty}(x_n) = \sup\{x_n \mid n \in \mathbb{N}\}$$

where sup is the supremum (Least Upper Bound) of the set.

What does this proposition mean?

Means that if a given sequence (x_n) is bounded and increasing then it converges to the supremum (Least Upper Bound) of the set consisting of the sequence (x_n).

Proof. Given above in the proof of Monotone Convergence Theorem (5.23).

Proposition (5.25). If (x_n) is a bounded decreasing sequence of real numbers then

$$\lim_{n\to\infty}(x_n) = \inf\{x_n \mid n \in \mathbb{N}\}$$

where inf is the infimum (Greatest Lower Bound) of the set.

What does this proposition mean?

Means that if a given sequence (x_n) is bounded and decreasing then it converges to the infimum (Greatest Lower Bound) of the set consisting of the sequence (x_n).

Proof. Exercise 5(f).

Example 25

Let (x_n) be a real sequence defined by

$$x_1 = 1 \quad \text{and} \quad x_{n+1} = \sqrt{x_n} \qquad (\dagger)$$

Show that this sequence (x_n) converges to 1.

Proof. We need to show that the given sequence (x_n) is a bounded and monotonic sequence. We can evaluate the first few terms of the sequence (x_n) by substituting $n = 1,\ 2,\ 3,\ 4, \cdots$ into (†)

$x_1 = 1$ [Given]

$x_2 = \sqrt{x_1} = \sqrt{1} = 1$ [Substituting $n = 1$ and $x_1 = 1$]

$x_3 = \sqrt{x_2} = \sqrt{1} = 1$ [Substituting $n = 2$ and $x_2 = 1$]

$x_4 = \sqrt{x_3} = \sqrt{1} = 1$ [Substituting $n = 3$ and $x_3 = 1$]

Hence $x_1 = x_2 = x_3 = x_4 = 1$. By induction we show that for all $n \in \mathbb{N}$

$$x_n = 1$$

Check the result for $n = 1$:

$$x_1 = 1 \quad \surd$$

Assume the result is true for $n = k$:

$$x_k = 1$$

Required to prove the result for $n = k + 1$:
By (†) we have

$$x_{k+1} = \sqrt{x_k} = \sqrt{1} = 1$$

Hence by induction we have for all $n \in \mathbb{N}$

$$x_n = 1$$

This means that the sequence $x_n = 1,\ 1,\ 1,\ 1,\ \cdots$

Therefore the given sequence (x_n) is bounded by 1 and monotonic (increasing) because for all $n \in \mathbb{N}$ we have

$1 \le 1 \le 1 \le 1 \le 1 \cdots$ which means that $x_1 \le x_2 \le x_3 \le \cdots \le x_n \le x_{n+1} \cdots$

By the Monotone Convergence Theorem (5.23) we conclude that the given sequence (x_n) converges. *What is the value of* $\lim_{n\to\infty}(x_n)$*?*

By proposition (5.24) we have

$$\lim_{n\to\infty}(x_n) = \sup\{x_n \mid n \in \mathbb{N}\}$$
$$= \sup\{1\} \qquad \text{[Because } x_n = 1 \text{ for all } n \in \mathbb{N}]$$
$$= 1$$

We have shown that the given sequence

$$x_1 = 1 \text{ and } x_{n+1} = \sqrt{x_n}$$

converges to 1.

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Example 26

Let (x_n) be a real sequence defined by

$$x_1 = 1 \text{ and } x_{n+1} = \frac{x_n}{2}$$

Show that this sequence (x_n) is bounded, monotonic and converges to 0.

Proof. We need to show that the given sequence (x_n) is a bounded and monotonic sequence and then by Monotone Convergence Theorem (5.23) we conclude that the sequence (x_n) converges.

We can evaluate the first few terms of the given sequence (x_n) by substituting $n = 1,\ 2,\ 3,\ 4, \cdots$ into $x_{n+1} = \frac{x_n}{2}$:

$$x_1 = 1 \qquad [\text{Given}]$$

$$x_2 = \frac{x_1}{2} = \frac{1}{2} \qquad [\text{Substituting } n = 1 \text{ and } x_1 = 1]$$

$$x_3 = \frac{x_2}{2} = \frac{1/2}{2} = \frac{1}{4} \qquad [\text{Substituting } n = 2 \text{ and } x_2 = 1/2]$$

$$x_4 = \frac{x_3}{2} = \frac{1/4}{2} = \frac{1}{8} \qquad [\text{Substituting } n = 3 \text{ and } x_3 = 1/4]$$

It looks as if the sequence (x_n) is bounded below by 0. *What do we need to show?*
Need to show that for all $n \in \mathbb{N}$ we have the inequality $x_n \le 1$. *How?*
We use induction.
Check the result for $n = 1$, that is $x_1 = 1 \le 1$ √
Assume the result is true for $n = k$ that is

$$x_k \le 1$$

Consider $n = k + 1$:

$$x_{k+1} = \frac{x_k}{2} \le 1 \qquad [\text{Because } x_k \le 1]$$

Hence by induction we have $x_n \le 1$ for all $n \in \mathbb{N}$ which means (x_n) is a bounded sequence.
The first few terms of the sequence are $1,\ 1/2,\ 1/4,\ 1/8, \cdots$ which suggests it is a decreasing sequence. Actually we can show that the given sequence (x_n) is a decreasing (monotonic) sequence. *How?*
Again by induction. We have

$$x_2 = 1/2 < 1 = x_1 \qquad √$$

Therefore $x_2 < x_1$. Assume $x_k \le x_{k-1}$. Required to prove the result $x_{k+1} \le x_k$.

$$x_{k+1} = \frac{x_k}{2} \le x_k$$

Therefore by induction we conclude that (x_n) is a decreasing (monotonic) sequence.
Hence we have shown that the given sequence is bounded and monotone and therefore by Monotone Convergence Theorem (5.23) we conclude that the given sequence (x_n) converges. *How do we show* $\lim_{n\to\infty}(x_n) = 0$ *?*
By the above:
Proposition (5.25). If (x_n) is a bounded decreasing sequence of real numbers then

$$\lim_{n\to\infty}(x_n) = \inf\{x_n \mid n \in \mathbb{N}\}$$

We have

$$\lim_{n\to\infty}(x_n) = \inf\{x_n \mid n \in \mathbb{N}\}$$

$$= \inf\left\{1,\ \frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ \frac{1}{16},\ \frac{1}{32}, \cdots\right\} = 0$$

Hence $\lim_{n\to\infty}(x_n)=0$.

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The sequences in Example 25 and 26 are said to be defined recursively.
What does a sequence defined recursively mean?
A sequence (x_n) is defined recursively if

i. $x_1 = a$ is given
ii. $x_{n+1} = f(x_n)$ the term x_{n+1} in the sequence (x_n) is a function of the previous term x_n

The sequence defined in Example 26 above is an example of a recursive sequence:

$$x_1 = 1 \quad \text{and} \quad x_{n+1} = \frac{x_n}{2}$$

For sequences which are defined recursively such as

$$x_1 = a, \qquad x_{n+1} = f(x_n)$$

we can use the following proposition (5.26) to evaluate its limit.

Proposition (5.26)
If $\lim_{n\to\infty}(x_n)=L$ then $\lim_{n\to\infty}(x_{n+1})=L$.

Proof. Since $\lim_{n\to\infty}(x_n)=L$ therefore by the definition of limit of a sequence (5.11) we have for an arbitrary $\varepsilon>0$ a number N_0 such that for all $n>N_0$

$$|x_n - L| < \varepsilon$$

Therefore $n+1>n>N_0$ we have

$$|x_{n+1} - L| < \varepsilon$$

This means that $\lim_{n\to\infty}(x_{n+1})=L$.

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We use this proposition to evaluate the limit of a sequence that is defined recursively.

Example 27

Let (x_n) be a sequence of real numbers defined recursively by

$$x_1 = 2 \quad \text{and} \quad x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n} \qquad (☺)$$

Prove that the sequence (x_n) defined in (☺) is decreasing and for all $n\in\mathbb{N}$, $x_n \ge \sqrt{2}$.
Also prove that $\lim_{n\to\infty}(x_n)=\sqrt{2}$.
[Hint: $a^2+b^2\ge 2ab$]

Proof. We can evaluate the first few terms of the sequence (x_n) by substituting $n=1,\ 2,\ 3,\ 4,\ \cdots$ into (☺) and working to 4 decimal places:

(5.11) $\lim_{n\to\infty}(x_n)=L \iff \forall\varepsilon>0 \ \ \exists N_0$ such that $\forall n>N_0$ $|x_n - L|<\varepsilon$

$x_1 = 2$ [Given]

$x_2 = \frac{x_1}{2} + \frac{1}{x_1} = \frac{2}{2} + \frac{1}{2} = 1.5000$ [Substituting $n = 1$ and $x_1 = 2$]

$x_3 = \frac{x_2}{2} + \frac{1}{x_2} = \frac{1.5}{2} + \frac{1}{1.5} = 1.4166$ [Substituting $n = 2$ and $x_2 = 1.5$]

$x_4 = \frac{x_3}{2} + \frac{1}{x_3} = \frac{1.4166}{2} + \frac{1}{1.4166} = 1.4142$ [Substituting $n = 3$ and $x_3 = 1.4166$]

We show that for all $n \in \mathbb{N}$

$$x_n \geq \sqrt{2}$$

By (☺) we have

$$x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}$$

$$= \frac{(x_n)^2 + 2}{2x_n} \quad \text{[Common Denominator]}$$

$$= \frac{(x_n)^2 + (\sqrt{2})^2}{2x_n} \quad \left[\text{Rewriting } 2 = (\sqrt{2})^2\right]$$

$$\geq \frac{2x_n\sqrt{2}}{2x_n} \quad \left[\begin{array}{l}\text{Using Hint } a^2 + b^2 \geq 2ab \\ \text{on Numerator } (x_n)^2 + (\sqrt{2})^2 \geq 2x_n\sqrt{2}\end{array}\right]$$

$$= \sqrt{2} \quad \text{[Cancelling Out Common Factors } 2x_n\text{]}$$

Hence $x_{n+1} \geq \sqrt{2}$. Since $x_1 = 2 \geq \sqrt{2}$, then for all $n \in \mathbb{N}$

$$x_n \geq \sqrt{2} \qquad (*)$$

The given sequence (x_n) is bounded.

The first few terms of the sequence are 2, 1.5, 1.4166, 1.4142, …We show that the given sequence (x_n) is decreasing (monotonic) which means that for all $n \in \mathbb{N}$ we have

$$x_{n+1} \leq x_n$$

How do we show this result?

We use (*) and show that $x_{n+1} - x_n \leq 0$.

$$x_{n+1} - x_n = \underbrace{\frac{x_n}{2} + \frac{1}{x_n}}_{= x_{n+1}} - x_n$$

$$= \frac{(x_n)^2 + 2 - 2(x_n)^2}{2x_n} \quad \text{[Common Denominator]}$$

$$= \frac{2 - (x_n)^2}{2x_n} \quad \text{[Simplifying]}$$

$$\leq 0 \quad \left[\begin{array}{l}\text{Because by (*) we have} \\ x_n \geq \sqrt{2} \text{ which implies } (x_n)^2 \geq 2\end{array}\right]$$

Hence $x_{n+1} - x_n \le 0$ which gives our required result $x_{n+1} \le x_n$ for all $n \in \mathbb{N}$.
Therefore the given sequence (x_n) is decreasing (monotonic). Since the given sequence (x_n) is bounded and monotonic therefore by the Monotone Convergence Theorem it converges.
Let $x = \lim_{n\to\infty}(x_n)$ then by the above proposition (5.26) we have $x = \lim_{n\to\infty}(x_{n+1})$. Using these to evaluate the limit of the given sequence in (☺):

$$x = \lim_{n\to\infty}(x_{n+1}) = \frac{\lim_{n\to\infty}(x_n)}{2} + \frac{1}{\lim_{n\to\infty}(x_n)} = \frac{x}{2} + \frac{1}{x}$$

Solving this equation we have

$$x = \frac{x}{2} + \frac{1}{x}$$

$$2x^2 = x^2 + 2 \qquad [\text{Multiplying by } 2x]$$

$$x^2 = 2 \qquad [\text{Simplifying}]$$

$$x = \pm\sqrt{2} \qquad [\text{Solving}]$$

$$x = \sqrt{2}, \quad x = -\sqrt{2}$$

$x = \lim_{n\to\infty}(x_n) = \sqrt{2}$ or $x = \lim_{n\to\infty}(x_n) = -\sqrt{2}$. *But which one of these is the limiting value of the given sequence?*
Since $x = \lim_{n\to\infty}(x_n) \ge 0$ because $x_n \ge 0$ for all $n \in \mathbb{N}$, therefore $x = \lim_{n\to\infty}(x_n) = \sqrt{2}$.

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SUMMARY
We have proved the Monotone Convergence Theorem which says:
(5.23) A **bounded monotone** sequence of real numbers is **convergent**.
We can use the supremum (Least Upper Bound) and infimum (Greatest Lower Bound) to find the limit of sequences.
(5.24). If (x_n) is a **bounded increasing** sequence of real numbers then

$$\lim_{n\to\infty}(x_n) = \sup\{x_n \mid n \in \mathbb{N}\}$$

(5.25). If (x_n) is a **bounded decreasing** sequence of real numbers then

$$\lim_{n\to\infty}(x_n) = \inf\{x_n \mid n \in \mathbb{N}\}$$

If a sequence is defined recursively then we can use the following result to evaluate the limit.

$$\lim_{n\to\infty}(x_n) = \lim_{n\to\infty}(x_{n+1})$$