Section C **The Limit of a Sequence**

By the end of this section you will be able to
- understand the definition of a limit of a sequence
- apply the definition to prove the given limit of a sequence
- apply inequalities and properties of the modulus function to prove the limit of a sequence

C1 **Revision of Inequalities and Modulus Function**

In this section you need to know the rules of inequalities and properties of the modulus function. This section is particularly difficult because you need to apply the rules of inequalities and properties of the modulus function to unfamiliar ground. One inequality which is used throughout this section is the following:

$$\text{If } a > b \text{ then } \frac{1}{a} < \frac{1}{b} \qquad (*)$$

That is if a is greater than b then the reciprocal $\frac{1}{a}$ is less than $\frac{1}{b}$. For example 3>2 but $\frac{1}{3} < \frac{1}{2}$. Note that the inequality changes when you take the reciprocal.

Another example is, let $n \in \mathbb{N}$ then $n+1 > n$ which also implies from the above inequality rule (*) that $\frac{1}{n+1} < \frac{1}{n}$. This example says, let n be a natural number (or a positive integer) then $n+1$ is greater than n which implies that the reciprocal $\frac{1}{n+1}$ is less than $\frac{1}{n}$.

Let n and N_0 both be natural numbers. *If* $n > N_0$ *then what inequality relates* $\frac{1}{n}$ *and* $\frac{1}{N_0}$*?*

Clearly applying the above inequality rule (*) we have

$$\frac{1}{n} < \frac{1}{N_0} \qquad \text{[Inequality Changes]}$$

If $n+1 > n > N_0$ *then what inequality relates* $\frac{1}{n+1}$, $\frac{1}{n}$ *and* $\frac{1}{N_0}$*?*

$$\frac{1}{n+1} < \frac{1}{n} < \frac{1}{N_0} \qquad \text{[Inequality Changes]}$$

You need to be comfortable in applying these inequalities because we will be using these throughout this section.

You also need to know the properties of the modulus function (sometimes called the distance function) such as if n is a natural number then

$$|n| = n$$

because n is positive. Similarly $|n+1| = n+1$ and so on. *What is* $|-1|$ *equal to?*

By the definition of the modulus function:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

we have $|-1| = 1$. *What is* $|-5|$ *equal to?*

$|-5| = 5$. *What is* $|3|$ *equal to?*

$|3| = 3$. Remember the modulus function is always positive or zero. We also use the following property:

$$\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$$

What is the difference between the Left Hand and Right Hand Side of the equals sign?
The modulus of a divided by b is equal to the modulus of a divided by the modulus of b.
We use the Greek symbol epsilon, ε, to denote a small positive real number in the remaining part of this section. It is normal practice in mathematical analysis to let ε be a small positive real number. Don't let this symbol confuse you.

C2 **Example of a Limit of a Sequence**

Example 8

Assume

$$\lim_{n\to\infty}\left(\frac{2n+5}{n+2}\right) = 2 \qquad \text{where } n \in \mathbb{N}$$

Let the symbol ε be an arbitrary positive real number. Find a number N_0 such that for all $n > N_0$ the following inequality holds:

$$\left|\frac{2n+5}{n+2} - 2\right| < \varepsilon$$

This inequality says "can we make the distance between $\frac{2n+5}{n+2}$ and 2 as small (ε) as we please for large enough n." Remember ε is any positive number.

Solution

What do we need to find?

Determine a number N_0 such that for all $n > N_0$ we have the following inequality:

$$\left|\frac{2n+5}{n+2} - 2\right| < \varepsilon \qquad \text{[Any Positive Number]}$$

Expanding the Left Hand Side of this inequality we have

$$\left|\frac{2n+5}{n+2}-2\right| = \left|\frac{2n+5-2(n+2)}{n+2}\right| \qquad \text{[Common Denominator]}$$

$$= \left|\frac{2n+5-2n-4}{n+2}\right| \qquad \text{[Expanding Numerator]}$$

$$= \left|\frac{1}{n+2}\right| \qquad \text{[Simplifying Numerator]}$$

$$= \frac{|1|}{|n+2|} = \frac{1}{n+2} \qquad \left[\text{Remember } \left|\frac{a}{b}\right| = \frac{|a|}{|b|}\right]$$

Since $n+2>n$ therefore $\frac{1}{n+2}<\frac{1}{n}$ by (*). Similarly we are told that $n>N_0$ which implies $\frac{1}{n}<\frac{1}{N_0}$. Combining these inequalities gives $\frac{1}{n+2}<\frac{1}{n}<\frac{1}{N_0}$ and so from above we have

$$\left|\frac{2n+5}{n+2}-2\right| = \frac{1}{n+2} < \frac{1}{N_0} = \varepsilon$$

We let $\frac{1}{N_0}=\varepsilon$ because we need the inequality $\left|\frac{2n+5}{n+2}-2\right|<\varepsilon$. *What is the value of the positive real number N_0?*

Since $\frac{1}{N_0}=\varepsilon$ transposing gives $N_0=\frac{1}{\varepsilon}$.

Hence we have for $N_0=\frac{1}{\varepsilon}$ and for all $n>N_0$ the following inequality holds:

$$\left|\frac{2n+5}{n+2}-2\right| < \varepsilon \quad \text{[Any Positive Number]}$$

This means that we can make the distance between $\frac{2n+5}{n+2}$ and 2 arbitrary small by considering an appropriate real number N_0. For example if we want the distance between $\frac{2n+5}{n+2}$ and 2 to be less than $\varepsilon=0.05$ then we take $N_0=\frac{1}{\varepsilon}=\frac{1}{0.05}=20$. We can check this as follows:

Substituting for $n>N_0=20$, that is $n=21$ say, into $\left|\frac{2n+5}{n+2}-2\right|$ we have

$$\left|\frac{(2\times 21)+5}{21+2}-2\right| = \left|\frac{47}{23}-2\right| = \left|\frac{1}{23}\right| = 0.043 < 0.05 = \varepsilon$$

This means that we have to consider more than the first 20 terms ($n=21$) of the

If $a>b$ then $\frac{1}{a}<\frac{1}{b}$ (*)

sequence so that the difference between $\dfrac{2n+5}{n+2}$ and 2 is less than $\varepsilon = 0.05$.

What is the value of positive number N_0 for $\varepsilon = 0.001$?

Substituting $\varepsilon = 0.001$ into $N_0 = \dfrac{1}{\varepsilon} = \dfrac{1}{0.001} = 1000$. This means that we need to consider more than first 1000 terms of the sequence for the distance between $\dfrac{2n+5}{n+2}$ and 2 to be less than $\varepsilon = 0.001$. For example let $n = 1001$, the first term after a thousand, then substituting this into the above gives

$$\left|\frac{2n+5}{n+2} - 2\right| = \left|\frac{(2\times 1001)+5}{1001+2} - 2\right| = 0.000\ 997 < 0.001$$

What is the value of N_0 for $\varepsilon = 1\times 10^{-6}$?

Using $N_0 = \dfrac{1}{\varepsilon}$ we have

$$N_0 = \frac{1}{\varepsilon} = \frac{1}{1\times 10^{-6}} = 1\times 10^{6} \quad \text{[One Million]}$$

Hence to make the distance between $\dfrac{2n+5}{n+2}$ and 2 to be less than 1×10^{-6} we need to consider terms after the first million (1 000 000). For example let $n = 1\ 000\ 001$, the first term after a million, then substituting this into the above gives

$$\left|\frac{2n+5}{n+2} - 2\right| = \left|\frac{(2\times 1\,000\,001)+5}{1\,000\,001+2} - 2\right| = 0.999\ 997\times 10^{-6} < 1\times 10^{-6}$$

Putting all these values in a table we have:

ε	$(n > N_0)$
0.05	21
0.0001	1001
1×10^{-6}	1 000 001

TABLE 1

We can always find a value of the natural number $n\ (> N_0)$ which gives the distance between $\dfrac{2n+5}{n+2}$ and 2 as small (ε) as we please.

We use this concept of the distance between the sequence (x_n) and the limit L to be small as we please for large enough n to define the limit of a sequence.

C3 **Definition**

Let (x_n) be a sequence of real numbers. Then the sequence (x_n) converges to limit L if and only if for every given positive real number ε there exists a positive real number N_0 such that for all $n > N_0$ we have

(5.11) $$|x_n - L| < \varepsilon \qquad (n \in \mathbb{N})$$

In notation form we can write this as:

$\lim_{n\to\infty}(x_n) = L \quad \Leftrightarrow \quad \forall \varepsilon > 0 \ \exists N_0 \in \mathbb{R}^+$ such that $\forall n > N_0$

(5.11) $$|x_n - L| < \varepsilon$$

What does this mean?

$|x_n - L| < \varepsilon$ means the distance between x_n and L is less than ε. That is the sequence x_n lies between $L-\varepsilon$ to $L+\varepsilon$.

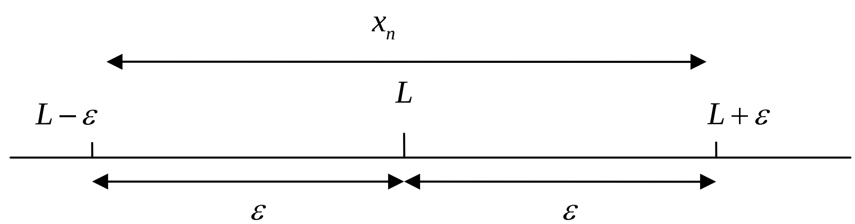

Fig 16

In Example 8, $x_n = \dfrac{2n+5}{n+2}$, $L = 2$ and we proved $\lim_{n\to\infty}\left(\dfrac{2n+5}{n+2}\right) = 2$.

For explanation purposes let N_0 be a natural number. Then for all $n > N_0$, $|x_n - L| < \varepsilon$ means that after N_0 terms of the sequence (x_n), all the elements x_{N_0+1}, x_{N_0+2}, x_{N_0+3}, ... all lie within ε distance of L.

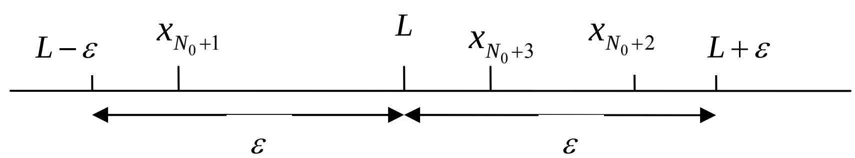

Fig 17

We apply this definition (5.11) to prove that a sequence has a particular limit. We are given a positive real number ε and we need to find a positive number N_0 which depends on ε. In many textbooks on mathematical analysis the number N_0 is normally defined to be a natural number. It makes the algebra of inequalities more difficult with N_0 being a natural number so for the time being we will take N_0 to be a positive real number.

It was the French mathematician, Cauchy, who gave us this definition of a limit. He was the first mathematician who made mathematics rigorous.

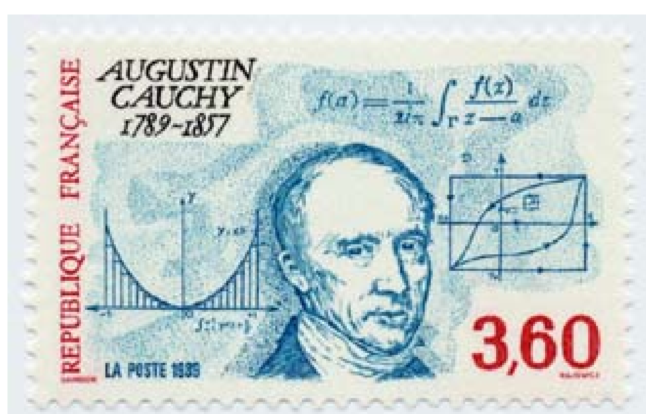

Fig 18

Augustin Cauchy 1789 to 1857.

Cauchy was born in Paris in 1789, the year of the French Revolution. In 1805 he passed the entrance exam for the prestigious Ecole Polytechnique and graduated in 1807. After graduating Cauchy went into the field of engineering.
However he returned to Ecole Polytechnique in 1813 to take up mathematics again.
The great mathematician Abel said of Cauchy:
"He is the only one who knows how mathematics should be done."
In 1831 Cauchy took up the chair in mathematical physics in Turin because his strong Catholic beliefs were causing him problems in France.
Cauchy was one of the first persons to put rigour into mathematics. He produced a text called 'Cours d' Analyse' in 1821 for a course in analysis at the Ecole Polytechnique which contained rigorous definitions of convergence, differentiability, integrability etc.

C4 **Examples**

Example 9

Prove that

$$\lim_{n\to\infty}\left(\frac{1}{n}\right)=0$$

Proof. We use the above definition

(5.11) $$|x_n - L| < \varepsilon$$

with $x_n = \frac{1}{n}$ and $L = 0$ to prove the given limit. To prove this limit we have to find a number N_0 such that for all $n > N_0$ the inequality (5.11) holds.

Let $\varepsilon > 0$ be given, then there is a positive real number N_0 such that for all $n > N_0$

$$\left|\frac{1}{n} - 0\right| = \left|\frac{1}{n}\right|$$

$$= \frac{1}{n} < \frac{1}{N_0} = \varepsilon \qquad \left[\text{Remember } n > N_0 \text{ implies } \frac{1}{n} < \frac{1}{N_0}\right]$$

We let $\frac{1}{N_0} = \varepsilon$ because we want the inequality $\left|\frac{1}{n} - 0\right| < \varepsilon$. *What is N_0 equal to?*

Transposing $\frac{1}{N_0} = \varepsilon$ gives $N_0 = \frac{1}{\varepsilon}$. Since we have found a real number N_0 such that $\forall n > N_0$ we have $\left|\frac{1}{n} - 0\right| < \varepsilon$ then this proves $\lim_{n\to\infty}\left(\frac{1}{n}\right) = 0$.

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If you can follow Example 9 then the remaining examples should be straightforward. Only the algebra is more challenging but the procedure is the same.
If we can find a positive real number N_0 such that $\forall n > N_0$ we have the following inequality

$$\left|x_n - L\right| < \varepsilon$$

for any given $\varepsilon > 0$, however small, then $\lim_{n\to\infty}(x_n) = L$.

The challenge is to find a number N_0 for every $\varepsilon > 0$. Note that when we say $\forall n > N_0$ we take a natural number n which is greater than N_0. For example if $N_0 = 120.32$ then we can consider $n = 121$.

Example 10

Prove that

$$\lim_{n\to\infty}\left(\frac{1}{n^2}\right) = 0$$

Proof. Again we need to establish the inequality in the above definition

(5.11) $$\left|x_n - L\right| < \varepsilon$$

What is x_n and L equal to in this case?

Remember x_n is the given sequence, $x_n = \frac{1}{n^2}$ and L the limiting value, $L = 0$. Let $\varepsilon > 0$ be given then there is a number N_0 such that $\forall n > N_0$

$$\left|\frac{1}{n^2} - 0\right| = \left|\frac{1}{n^2}\right|$$

$$= \frac{1}{n^2} < \frac{1}{N_0^{\;2}} = \varepsilon$$

Remember $n > N_0$ implies $n^2 > N_0^{\;2}$ therefore by (*) we have $\frac{1}{n^2} < \frac{1}{N_0^{\;2}}$.

We let $\frac{1}{N_0^{\;2}} = \varepsilon$ in the above because we want $\left|\frac{1}{n^2} - 0\right| < \varepsilon$. *What is N_0 equal to?*

Transposing $\frac{1}{N_0^{\;2}} = \varepsilon$ we have

$$N_0^{\;2} = \frac{1}{\varepsilon}$$

$$N_0 = \sqrt{\frac{1}{\varepsilon}} = \frac{\sqrt{1}}{\sqrt{\varepsilon}} = \frac{1}{\sqrt{\varepsilon}}$$

Since we have found a real number $N_0 = \frac{1}{\sqrt{\varepsilon}}$ such that for all $n > N_0$ we have $\frac{1}{n^2} < \varepsilon$ or $\left|\frac{1}{n^2} - 0\right| < \varepsilon$ then this proves $\lim_{n\to\infty}\left(\frac{1}{n^2}\right) = 0$.

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(*) $a > b \Rightarrow \frac{1}{a} < \frac{1}{b}$

Since in Example 10 we have shown that for every $\varepsilon > 0$ there is a number $N_0 = \dfrac{1}{\sqrt{\varepsilon}}$ such that for all $n > N_0$ we have the inequality $\left|\dfrac{1}{n^2} - 0\right| < \varepsilon$ therefore we have proven

$$\lim_{n\to\infty}\left(\frac{1}{n^2}\right) = 0$$

The next example is very similar to the above two but the algebra is more demanding.

Example 11

Prove that $\displaystyle\lim_{n\to\infty}\left(\frac{n-1}{n+1}\right) = 1$

Proof. To find N_0 we examine $\left|x_n - L\right|$ where x_n is the formula for the sequence and L is the limit. *What is x_n and L equal to?*

In this case $x_n = \dfrac{n-1}{n+1}$ and $L = 1$. Let $\varepsilon > 0$ be given, then there is a number N_0 such that $\forall n > N_0$ we have

$$\begin{aligned}
\left|\frac{n-1}{n+1} - 1\right| &= \left|\frac{n-1-(n+1)}{n+1}\right| && \text{[Common Denominator]} \\
&= \left|\frac{n-1-n-1}{n+1}\right| && \text{[Expanding Numerator]} \\
&= \left|\frac{-2}{n+1}\right| && \text{[Simplifying Numerator]} \\
&= \frac{|-2|}{n+1} \\
&= \frac{2}{n+1} < \frac{2}{n} < \frac{2}{N_0} = \varepsilon && \left[\begin{array}{l}\text{Because } n+1 > n > N_0 \\ \text{implies } \dfrac{1}{n+1} < \dfrac{1}{n} < \dfrac{1}{N_0}\end{array}\right]
\end{aligned}$$

Again we let $\dfrac{2}{N_0} = \varepsilon$ because we want the inequality $\left|\dfrac{n-1}{n+1} - 1\right| < \varepsilon$. *What is N_0 equal to?*

Transposing $\dfrac{2}{N_0} = \varepsilon$ gives $N_0 = \dfrac{2}{\varepsilon}$.

Since we have found a positive real number $N_0 = \dfrac{2}{\varepsilon}$ such that for all $n > N_0 = \dfrac{2}{\varepsilon}$ the inequality $\left|\dfrac{n-1}{n+1} - 1\right| < \varepsilon$ holds, therefore this proves $\displaystyle\lim_{n\to\infty}\left(\frac{n-1}{n+1}\right) = 1$.

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Example 12

Prove that

$$\lim_{n\to\infty}\left(\frac{3n+1}{4n+2}\right)=\frac{3}{4}$$

Proof. Again the challenge is to find an appropriate value of N_0 so that for all $n > N_0$ we have the inequality

$$|x_n - L| < \varepsilon$$

To find the number N_0 we examine $|x_n - L|$ where $x_n = \dfrac{3n+1}{4n+2}$ and $L = \dfrac{3}{4}$ is the limit. Let $\varepsilon > 0$ be given, then there is a real number N_0 such that $\forall n > N_0$ we have

$$\begin{aligned}
\left|\frac{3n+1}{4n+2}-\frac{3}{4}\right| &= \left|\frac{4(3n+1)-3(4n+2)}{4(4n+2)}\right| && \left[\text{Common Denominator}\right]\\
&= \frac{|12n+4-12n-6|}{|4(4n+2)|} && \left[\text{Expanding Numerator}\right]\\
&= \frac{|-2|}{|4(4n+2)|} && \left[\text{Simplifying Numerator}\right]\\
&= \frac{2}{4(4n+2)} && \left[\text{Remember } |-2| = 2\right]\\
&= \frac{1}{2(4n+2)} && \left[\text{Because } \frac{2}{4}=\frac{1}{2}\right]\\
&= \frac{1}{8n+4} < \frac{1}{8n} < \frac{1}{8N_0} = \varepsilon && \left[\begin{array}{l}\text{Using rules of}\\ \text{Inequalities}\end{array}\right]
\end{aligned}$$

The last line follows from:

$$8n+4 > 8n > 8N_0 \text{ which implies } \frac{1}{8n+4} < \frac{1}{8n} < \frac{1}{8N_0}$$

Transposing $\dfrac{1}{8N_0} = \varepsilon$ gives $N_0 = \dfrac{1}{8\varepsilon}$. Since $\forall \varepsilon > 0$ we have found a number $N_0\left(=\dfrac{1}{8\varepsilon}\right)$ such that $\forall n > N_0$

$$\left|\frac{3n+1}{4n+2}-\frac{3}{4}\right| < \varepsilon$$

therefore we have proven $\lim\limits_{n\to\infty}\left(\dfrac{3n+1}{4n+2}\right)=\dfrac{3}{4}$. ■

Example 13

Prove that

$$\lim_{n\to\infty}\left(\frac{2n+7}{n-3}\right)=2$$

Proof. Let $\varepsilon > 0$ be given. We need to find a real number N_0 such that $\forall n > N_0$ we have

$$\left|\frac{2n+7}{n-3}-2\right| < \varepsilon$$

How to we find this number N_0 *?*

Examining the Left Hand Side of this inequality we have $\forall n > N_0$

$$\begin{aligned}\left|\frac{2n+7}{n-3}-2\right| &= \left|\frac{2n+7-2(n-3)}{n-3}\right| \\ &= \left|\frac{2n+7-2n+6}{n-3}\right| \\ &= \left|\frac{13}{n-3}\right| = \frac{13}{n-3} < \frac{13}{N_0-3} = \varepsilon\end{aligned}$$

The last line, $\frac{13}{n-3} < \frac{13}{N_0-3}$, follows from

$$n > N_0 \text{ implies } n-3 > N_0-3 \text{ which implies } \frac{1}{n-3} < \frac{1}{N_0-3}$$

How do we determine the number N_0 *?*

From above we have

$$\frac{13}{N_0-3} = \varepsilon$$

$$\frac{13}{\varepsilon} = N_0 - 3 \text{ gives } \frac{13}{\varepsilon} + 3 = N_0 \text{ or written the other way } N_0 = \frac{13}{\varepsilon} + 3$$

Hence for any given $\varepsilon > 0$ and for all $n > N_0 = \frac{13}{\varepsilon} + 3$ we have

$$\left|\frac{2n+7}{n-3}-2\right| < \varepsilon$$

this proves $\lim_{n\to\infty}\left(\frac{2n+7}{n-3}\right) = 2$.

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Example 14

Prove that

$$\lim_{n\to\infty}\left(\frac{1}{3^n}\right) = 0$$

Proof. We need to find a positive real number N_0 such that $\forall n > N_0$

$$\left|\frac{1}{3^n} - 0\right| < \varepsilon$$

for any given $\varepsilon > 0$. *How to we find this number* N_0 *?*

Examining the Left Hand Side we have $\forall n > N_0$

$$\left|\frac{1}{3^n}-0\right|=\left|\frac{1}{3^n}\right|$$

$$=\frac{1}{3^n}<\frac{1}{3^{N_0}}=\varepsilon \qquad \left[\begin{array}{l}\text{Because } n>N_0 \text{ implies } 3^n>3^{N_0} \\ \text{which gives } \dfrac{1}{3^n}<\dfrac{1}{3^{N_0}}\end{array}\right]$$

How do we determine the number N_0 *from* $\frac{1}{3^{N_0}}=\varepsilon$ *?*

$$3^{-N_0}=\varepsilon \qquad \left[\text{Using the rules of indices to rewrite } \frac{1}{3^{N_0}}=3^{-N_0}\right]$$

$$\ln\left(3^{-N_0}\right)=\ln(\varepsilon) \qquad [\text{Taking Natural logs}]$$

$$-N_0\ln(3)=\ln(\varepsilon) \qquad \left[\text{Applying } \ln\left(a^n\right)=n\ \ln(a) \text{ on LHS}\right]$$

$$-N_0=\left[\frac{\ln(\varepsilon)}{\ln(3)}\right]$$

$$N_0=-\left[\frac{\ln(\varepsilon)}{\ln(3)}\right] \qquad [\text{Multiplying through by } -1\ ]$$

Hence for any given $\varepsilon>0$ and for all $n>N_0=-\left[\frac{\ln(\varepsilon)}{\ln(3)}\right]$ we have

$$\left|\frac{1}{3^n}-0\right|<\varepsilon$$

Remember N_0 needs to be positive. *But isn't the number* N_0 *negative in this example?* N_0 is only negative if $\varepsilon>1$ that means

$$\left|\frac{1}{3^n}-0\right|=\frac{1}{3^n}<\varepsilon$$

and $\frac{1}{3^n}<1$ is true for $\forall n\in\mathbb{N}$. Hence we have proven $\lim\limits_{n\to\infty}\left(\frac{1}{3^n}\right)=0$.

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SUMMARY

We use the formal definition of a limit of a sequence to prove the limits of sequences.

Let (x_n) be a sequence of real numbers then

$$\lim_{n\to\infty}(x_n)=L$$

$\Leftrightarrow\ \forall\varepsilon>0\ \exists N_0\in\mathbb{R}^+$ such that $\forall n>N_0$

$$|x_n-L|<\varepsilon \qquad (5.11)$$

The challenge is to find a positive real number N_0 such that for any given positive ε, inequality (5.11) holds.