SECTION C **IMPLICATION**

By the end of this section you will be able to

- state the truth table for implication
- identify a contradiction and a tautology
- construct the converse of implication
- obtain the contrapostive form of an implication

C1 **Truth Table of Implication**

We defined implication in section A of this chapter. Let P and Q be propositions then the compound proposition ‘P implies Q’ means ‘if P then Q’ and is denoted by $P \Rightarrow Q$. The truth table for P implies Q, $P \Rightarrow Q$, is given by:

	P	Q	$P \Rightarrow Q$
Row 1	T	T	T
Row 2	T	F	F
Row 3	F	T	T
Row 4	F	F	T

TABLE 11

You might think there is a misprint in TABLE 11, with regards to the bottom two rows, which is read as ‘if P is false’ then ‘$P \Rightarrow Q$’ is true, independent of the truth value of Q. There is **no** misprint, this is correct. *How can we justify these statements?*

Let’s consider an example where P and Q are the following propositions.

P : I am elected

Q : I will abolish the death penalty

If P is false that is ‘I am not elected’ then I am under no obligation to abolish the death penalty. Implication is like a contract or a promise.

The only situation when $P \Rightarrow Q$ is false (I have broken my promise) is

‘If I am elected then I do **not** abolish the death penalty.’

This situation is represented in Row 2 of TABLE 11.

In general the implication $P \Rightarrow Q$ is **only** false if P is true and Q is false otherwise $P \Rightarrow Q$ is true. It is critical that you realise there is a **difference** between factual and logical truth.

We can show that $P \Rightarrow Q$ is equivalent to $(\neg P) \vee Q$ that is (not P) or Q.

Example 20

Show that

$$(P \Rightarrow Q) \equiv ((\neg P) \vee Q) \qquad \text{[Equivalent]}$$

We have placed brackets on the left and right of the equivalent sign, $\equiv$, so that it becomes easier to visualize the propositions.

Solution

What does equivalence mean in this example?

It means $(\neg P)\vee Q$ and $P\Rightarrow Q$ have the same truth values for all possible combinations of truth values of P and Q.

In the first two left hand columns we list the combinations of truth values of P and Q in TABLE 12. The truth value of $P\Rightarrow Q$ is given in column 3 of the above TABLE 11. We can work out the truth value of $(\neg P)\vee Q$. *How?*

First determine the truth values of $\neg P$ (not P) and then $(\neg P)\vee Q$. Hence the truth table is:

Column 1	Column 2	Column 3	Column 4	Column 5
P	Q	$\neg P$	$(\neg P)\vee Q$	$P\Rightarrow Q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

TABLE 12

Since columns 4 and 5 agree for all possible combinations of truth values therefore they are equivalent, that is

$$(P\Rightarrow Q)\equiv((\neg P)\vee Q) \qquad \text{[Equivalent]}$$

Example 21

Construct a truth table for $(P\Rightarrow Q)\wedge(Q\Rightarrow R)$

Solution

We first list the combination of truth values for the propositions P, Q and R (in columns 1, 2 and 3 respectively). Next we evaluate

$P\Rightarrow Q$ (column 4) and $Q\Rightarrow R$ (column 5)

Finally we find the truth values of $(P\Rightarrow Q)\wedge(Q\Rightarrow R)$ in the right hand column.

Column 1	Column 2	Column 3	Column 4	Column 5	Column 6
P	Q	R	$P\Rightarrow Q$	$Q\Rightarrow R$	$(P\Rightarrow Q)\wedge(Q\Rightarrow R)$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	T	F
F	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T
F	F	F	T	T	T

TABLE 13

Next we define the logical terms tautology and contradiction.

C2 **Tautology**

Tautology is a compound proposition which is **always** true. A compound proposition which is always true may seem boring to investigate but that is the definition of tautology.

Example 22

Construct a truth table for $P \vee (\neg P)$.

Solution

This was question 3(b) of Exercise 1(b). Hence

P	$\neg P$	$P \vee (\neg P)$
T	F	T
F	T	T

TABLE 14

It doesn't matter what the truth value of P is but $P \vee (\neg P)$ is **always** true. This is sometimes written as

$$P \vee (\neg P) \equiv T$$

Hence $P \vee (\neg P)$ is an example of a tautology. This means P or (not P) is always true.

Let

P : London is the capital of New Zealand

Then the statement $P \vee (\neg P)$: 'London is the capital of New Zealand or it is not' is always true.

Let P be the statement '$x^2 - 9 = 0$' then '$\underbrace{x^2 - 9 = 0}_{P}$ or $\underbrace{x^2 - 9 \neq 0}_{\neg P}$' is **always** true.

That is $x^2 - 9$ is equal to zero or it is **not** equal to zero is an example of a tautology and therefore is always true.

Example 23

Construct a truth table for $\left[(P \Rightarrow Q) \wedge (Q \Rightarrow R)\right] \Rightarrow (P \Rightarrow R)$. *What do you notice about your result?*

Solution

We have already found the truth value of $\left[(P \Rightarrow Q) \wedge (Q \Rightarrow R)\right]$ in TABLE 13, column 6 above.

We can copy this into the table below and find the truth values of the remainder of the proposition.

P	Q	R	$[(P\Rightarrow Q)\wedge(Q\Rightarrow R)]$	$P\Rightarrow R$	$[(P\Rightarrow Q)\wedge(Q\Rightarrow R)]\Rightarrow(P\Rightarrow R)$
T	T	T	T	T	T
T	T	F	F	F	T
T	F	T	F	T	T
F	T	T	T	T	T
T	F	F	F	F	T
F	T	F	F	T	T
F	F	T	T	T	T
F	F	F	T	T	T

TABLE 15

The right hand column in TABLE 15 is obtained by finding the truth values of the 4th column $\Rightarrow$ 5th column.

By observing the right hand column of TABLE 15 we can say

$$[(P\Rightarrow Q)\wedge(Q\Rightarrow R)]\Rightarrow(P\Rightarrow R)$$

is a tautology. (It is always true). It can be written as

$$[(P\Rightarrow Q)\wedge(Q\Rightarrow R)]\Rightarrow(P\Rightarrow R)\equiv T$$

What does this proposition $[(P\Rightarrow Q)\wedge(Q\Rightarrow R)]\Rightarrow(P\Rightarrow R)$ *mean?*

It looks horrendous but all it says is

$$\text{‘If } \underbrace{P \text{ implies } Q \text{ and } Q \text{ implies } R}_{(P\Rightarrow Q)\wedge(Q\Rightarrow R)} \text{ then } \underbrace{P \text{ implies } R}_{P\Rightarrow R}\text{’.}$$

For example, let P, Q and R be the following propositions:

$$P:\ 2x+1=0,\quad Q:\ 2x=-1 \text{ and } R:\ x=-\frac{1}{2}$$

Using $[(P\Rightarrow Q)\wedge(Q\Rightarrow R)]\Rightarrow(P\Rightarrow R)$ we have

If

$$P\Rightarrow Q:\ 2x+1=0 \Rightarrow 2x=-1$$

and

$$Q\Rightarrow R:\ 2x=-1 \Rightarrow x=-\frac{1}{2}$$

then

$$P\Rightarrow R:\ 2x+1=0 \Rightarrow x=-\frac{1}{2}$$

Consider another example of the area of a circle with radius r. Let P, Q and R be the following propositions:

$$P:\ \pi r^2=A,\quad Q:\ r^2=\frac{A}{\pi} \text{ and } R:\ r=\sqrt{\frac{A}{\pi}}$$

If

$$P\Rightarrow Q:\ \pi r^2=A \Rightarrow r^2=\frac{A}{\pi}$$

and

$$Q \Rightarrow R : \quad r^2 = \frac{A}{\pi} \;\Rightarrow\; r = \sqrt{\frac{A}{\pi}}$$

Then

$$P \Rightarrow R : \quad \pi r^2 = A \quad \Rightarrow \quad r = \sqrt{\frac{A}{\pi}}$$

Remember (small) r is the radius of the circle and (capital) R is the given proposition.

C3 **Contradiction**

A compound proposition which is always false is called a **contradiction**.

Example 24

Construct a truth table for $P \wedge (\neg P)$. *What do you notice about your results?*

Solution

The truth table can be established as follows:

P	$\neg P$	$P \wedge (\neg P)$
T	F	F
F	T	F

TABLE 16

The right hand column of TABLE 16 shows that $P \wedge (\neg P)$ is a contradiction. In words this means P and (not P) is always false. Similarly you can show that $(\neg P) \wedge P$ is also a contradiction. (See Exercise).

Considering the above example, let

P : London is the capital of New Zealand

Then applying the proposition $P \wedge (\neg P)$ to this we have

'London is the capital of New Zealand' and 'London is **not** the capital of New Zealand' is false because London **cannot** be both the capital and **not** the capital of New Zealand.

Let P be the statement $x^2 - 1 = 0$. *What is $\neg P$ (not P) in this case?*

$$x^2 - 1 \neq 0$$

Hence $P \wedge (\neg P)$ is given by

$$x^2 - 1 = 0 \quad \text{and} \quad x^2 - 1 \neq 0$$

must **always** be false because you **cannot** have both

$$x^2 - 1 = 0 \left[\text{Equals Zero}\right] \text{ and } x^2 - 1 \neq 0 \left[\text{Not Equal to Zero}\right]$$

We say $x^2 - 1 = 0$ and $x^2 - 1 \neq 0$ is a contradiction.

C4 **Converse**

Let P and Q be propositions. We know implication between P and Q is denoted by $P \Rightarrow Q$. If we go the other way, which is $Q \Rightarrow P$ [*Q* implies *P*] then this is called the **converse** of $P \Rightarrow Q$ [*P* implies *Q*]. Let

P : I have two exotic holidays per year

Q : I am rich

What is $P \Rightarrow Q$ *?*

$$\text{If } \underbrace{\text{I have two exotic holidays per year}}_{P} \text{ then } \underbrace{\text{I am rich}}_{Q}.$$

What is $Q \Rightarrow P$ *?*

$$\text{If } \underbrace{\text{I am rich}}_{Q} \text{ then } \underbrace{\text{I have two exotic holidays per year}}_{P}.$$

Example 25

Construct a truth table for $Q \Rightarrow P$ and compare your solution with $P \Rightarrow Q$. *What do you notice?*

Solution

Remember $Q \Rightarrow P$ is **only** false when Q is true and P is false otherwise $Q \Rightarrow P$ is true. The truth table is given by

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

TABLE 17

What can you conclude about $P \Rightarrow Q$ *and* $Q \Rightarrow P$ *?*

Since the truth values of $P \Rightarrow Q$ and $Q \Rightarrow P$ do not match therefore we can conclude that $P \Rightarrow Q$ is **not** equivalent to $Q \Rightarrow P$:

$$(P \Rightarrow Q) \not\equiv (Q \Rightarrow P) \quad \text{[Not Equivalent]}$$

This is an **important** result which many students find difficult to accept especially when proving propositions. They sometimes prove the converse, $Q \Rightarrow P$ [Q implies P], and think they have proven $P \Rightarrow Q$ [P implies Q]. You need to be very careful.

Consider the example where P is the proposition '$x = 5$' and Q is the proposition '$x^2 = 25$' then $P \Rightarrow Q$ is given by:

$$x = 5 \quad \Rightarrow \quad x^2 = 25 \quad \text{which is true}$$

However the converse $Q \Rightarrow P$

$$x^2 = 25 \quad \Rightarrow \quad x = 5 \text{ is false because } x \text{ could equal } -5$$

Consider another example where P is the proposition 'a and b are odd' and Q is the proposition '$a + b$ is even' then $P \Rightarrow Q$ is given by:

$$a \text{ and } b \text{ are odd} \Rightarrow \quad a + b \text{ is even [This is true]}$$

But the converse $Q \Rightarrow P$:

$$a + b \text{ is even} \Rightarrow \quad a \text{ and } b \text{ are odd}$$

is FALSE. *Why?*

Consider $a + b = 6$ then a could be 4 and b could be 2. Hence $a + b$ is even but both a and b could also be even.

Another example is

$$x > 0 \text{ [Positive] and } y > 0 \text{ [Positive]} \quad \Rightarrow \quad xy > 0 \text{ [Positive]}$$

This is true. However the converse

$$xy > 0 \ [\text{Positive}] \quad \Rightarrow \quad x > 0 \ [\text{Positive}] \text{ and } y > 0 \ [\text{Positive}] \text{ is FALSE}$$

Why is this last proposition false?
Because
[Negative]× [Negative] is positive therefore from $xy > 0$ we could have both

$$x < 0 \,[\text{Negative}] \text{ and } y < 0 \,[\text{Negative}]$$

An example of where the converse is true is the following:

$$x - 5 < 0 \quad \Rightarrow \quad x < 5 \qquad [\text{This is true}]$$

$$x < 5 \quad \Rightarrow \quad x - 5 < 0 \quad [\text{This is also true}]$$

Hence the converse, $Q \Rightarrow P$ [Q implies P], of the proposition $P \Rightarrow Q$ [P implies Q] may or may **not** true.

C5 **Contrapositive**

Let P and Q be propositions then the **contrapositive** of $P \Rightarrow Q$ is the proposition $(\neg Q) \Rightarrow (\neg P)$ that is (not Q) implies (not P). Using the above example

P : I have two exotic holidays per year
Q : I am rich

What is the contrapositive of $P \Rightarrow Q$, *that is* $(\neg Q) \Rightarrow (\neg P)$, *for this example?*

$$\text{If } \underbrace{\text{I am not rich}}_{\neg Q} \text{ then } \underbrace{\text{I do not have two exotic holidays per year}}_{\neg P}.$$

Example 26

Construct a truth table for $(\neg Q) \Rightarrow (\neg P)$. *What do you notice about your answer in relation to the proposition* $P \Rightarrow Q$ *?*

Solution

We need to construct the truth table for (not Q) implies (not P), $(\neg Q) \Rightarrow (\neg P)$. In the first two left hand columns we list all the possible combinations of P and Q. For the next two columns we write down the truth tables for $\neg Q$ (not Q) and $\neg P$ (not P) respectively. In the right hand column we find the truth values of $(\neg Q) \Rightarrow (\neg P)$.

P	Q	$\neg Q$	$\neg P$	$(\neg Q) \Rightarrow (\neg P)$
T	T	F	F	T
T	F	T	F	F
F	T	F	T	T
F	F	T	T	T

TABLE 18

By comparing the right hand columns of TABLE 11 [Implication Table] and TABLE 18 we can say $(\neg Q) \Rightarrow (\neg P)$ and $P \Rightarrow Q$ are equivalent. That is

$$((\neg Q) \Rightarrow (\neg P)) \equiv (P \Rightarrow Q) \quad [\text{Equivalent}]$$

$(\neg Q) \Rightarrow (\neg P)$ has the same truth values as $P \Rightarrow Q$ for all combinations of truth values of P and Q.

This result, $((\neg Q)\Rightarrow(\neg P))\equiv(P\Rightarrow Q)$, is important and is often used to prove $P\Rightarrow Q$. That is if you prove $(\neg Q)\Rightarrow(\neg P)$ then you have proven $P\Rightarrow Q$.

These last two results are critical (and maybe hard to believe)

$$(Q\Rightarrow P)\not\equiv(P\Rightarrow Q) \quad \text{[Not Equivalent]}$$

$$((\neg Q)\Rightarrow(\neg P))\equiv(P\Rightarrow Q) \quad \text{[Equivalent]}$$

We say these two propositions maybe hard to believe because they tend to be against our intuition.

Example 27

Let ABC be a triangle with sides a, b and c where $a<b<c$. Pythagoras theorem states that:

If ABC is a right-angled triangle then

$$c^2=a^2+b^2$$

State the contrapositive and converse of Pythagoras theorem. Also state whether the converse is true or false.

Solution

Let P be the proposition 'ABC is a right-angled triangle' and Q be the proposition ' $c^2=a^2+b^2$ '. By the given statement of Pythagoras we have

$$P\Rightarrow Q$$

What is the converse of $P\Rightarrow Q$ *?*

It is $Q\Rightarrow P$ which means 'If $c^2=a^2+b^2$ then ABC is a right-angled triangle.'

The converse is also true. Generally Pythagoras theorem is given as $P\Rightarrow Q$ as in the above statement of this example. However $Q\Rightarrow P$ is also true and it does make us think whether the converse is true or not.

What is the contrapositive of $P\Rightarrow Q$ *?*

$(\neg Q)\Rightarrow(\neg P)$ which means 'If $c^2\neq a^2+b^2$ [Not Equal] then ABC is **not** a right-angled triangle.'

Example 28

Consider the proposition:

$$\text{If } x=y \text{ then } x^2=y^2$$

State the contrapositive and converse of this proposition. *Is the converse true or false?*

Solution

What is the contrapositive of

$$\textit{If } x=y \textit{ then } x^2=y^2 \textit{ ?}$$

In notation form this can be written as $x=y \Rightarrow x^2=y^2$. So the contrapositive of this is

$$x^2\neq y^2\Rightarrow x\neq y \quad \text{[Not Equal]}$$

The converse of $x=y \Rightarrow x^2=y^2$ is

$$x^2=y^2\Rightarrow x=y$$

Is this true?
No because it could be that $x = -y$ [Negative y] that is

$$x^2 = y^2 \Rightarrow \; x = -y$$

In the above examples why don't we ask the question '*Is the contrpositive true or false?*'
Because the contrapositive, $(\neg Q) \Rightarrow (\neg P)$, is equivalent to $P \Rightarrow Q$ and so has the same truth value as the original proposition, $P \Rightarrow Q$.
Note the important difference between the converse and contrapositive of a proposition.

SUMMARY
The implication is symbolized by $\Rightarrow$ and is placed between two propositions such as P and Q:

$$P \Rightarrow Q$$

This is **only** false if P is true and Q is false otherwise it is true.
A compound proposition which is always **true** is called a **tautology**.
A compound proposition which is always **false** is called a **contradiction**.
The **converse** of $P \Rightarrow Q$ is $Q \Rightarrow P$. Also it is important to note that

$$(Q \Rightarrow P) \not\equiv (P \Rightarrow Q) \quad \text{[Not Equivalent]}$$

The **contrapositive** of $P \Rightarrow Q$ is the proposition $(\neg Q) \Rightarrow (\neg P)$. More importantly they are equivalent that is

$$((\neg Q) \Rightarrow (\neg P)) \equiv (P \Rightarrow Q) \quad \text{[Equivalent]}$$