Section B: Uniform Convergence of functions

By the end of this section you will be able to

- understand what is meant by uniform convergence
- test a sequence for uniform convergence
- compare and contrast between uniform and pointwise convergence

B1 Uniform Convergence of Functions

In the last section we covered **pointwise** convergence. Remember pointwise convergence depends on the value of *x* in the domain of the function. However in uniform convergence the convergence is independent of *x*.

The sequence of functions given in Example 6 on page 9 demonstrates uniform convergence:

$$f_n(x) = \frac{\sin(nx)}{n}$$

Some of these functions are illustrated in Figure 11 below. The function $f_n(x)$ shown below is for $n \geq N_0$ where the natural number N_0 depends only on ε **not** *x*. Note that the sequence $f_n(x)$ for $n \geq N_0$ is within ε of the function $f(x)$ whatever the value of *x*. Compare this with Fig 9 on page 6 which shows pointwise convergence.

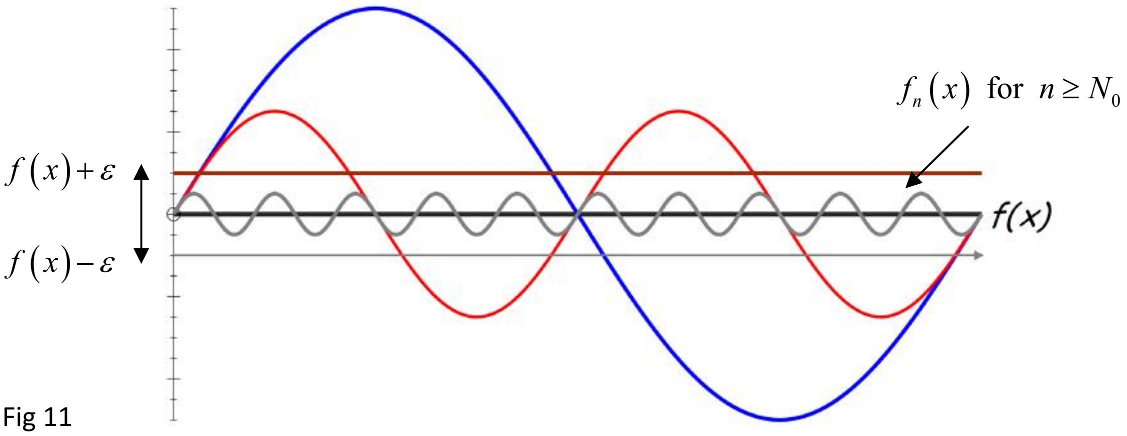

Fig 11

Example 9

Let $f_n : [0, \pi] \to \mathbb{R}$ be a sequence of functions given by:

$$f_n(x) = \frac{\sin(nx)}{n}$$

(i) Plot $f_1(x)$, $f_2(x)$, $f_3(x)$ and $f_{10}(x)$ on the same axes.

(ii) What is the value of $f_n(x)=\dfrac{\sin(nx)}{n}$ as $n\to\infty$?

Solution

(i) Substituting $n=1,\ 2,\ 3$ and 10 into $f_n(x)=\dfrac{\sin(nx)}{n}$ gives

$$f_1(x)=\frac{\sin(x)}{1}=\sin(x),\ f_2(x)=\frac{\sin(2x)}{2},\ f_3(x)=\frac{\sin(3x)}{3}\ \text{ and }\ f_{10}(x)=\frac{\sin(10x)}{10}$$

Plotting these gives

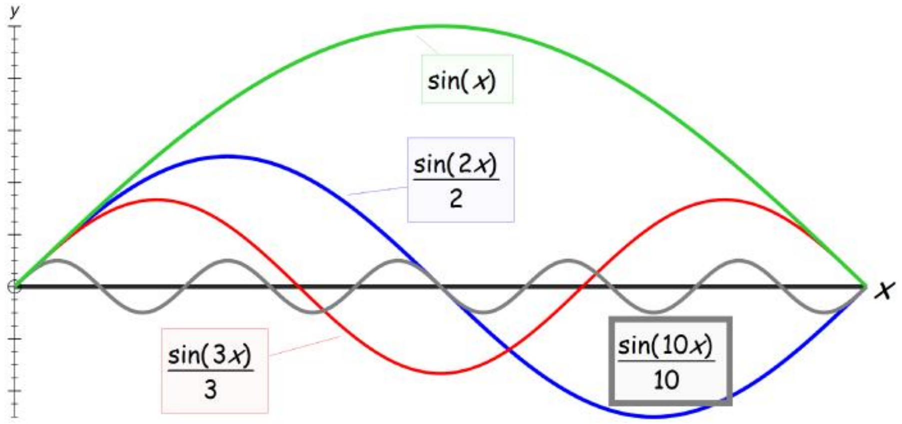

Fig 12

(ii) In Example 6 on page 7 we have already shown that this sequence of functions converges to the zero function, $f(x)=0$.

By Example 6 on page 9 the size of the natural number *n* for $f_n(x)$ such that $f_n(x)$ is within ε of zero (because these functions $f_n(x)=\dfrac{\sin(nx)}{n}$ converge to zero) is $n\ge N_0>\dfrac{1}{\varepsilon}$.

We examine some numerical values for the above example. *For* $\varepsilon=0.0001$ *what natural number* N_0 *ensures that* $f_n(x)=\dfrac{\sin(nx)}{n}$ *is within* $\varepsilon=0.0001$ *of zero?*

We need to find a N_0 such that for all $n\ge N_0$ we have

$$-0.0001<f_n(x)=\frac{\sin(nx)}{n}<0.0001$$

By the answer given in the above example we select our $N_0>\dfrac{1}{\varepsilon}=\dfrac{1}{0.0001}=10\,000$. Hence if our N_0 is greater than 10 000, $N_0=10\ 001$ say, then

$$-0.0001 < f_{10\,001}(x) = \frac{\sin(10001x)}{10001} < 0.0001$$

Remember the sine function lies between -1 and 1 therefore

$$-0.0001 < \frac{\pm 1}{10001} < 0.0001$$

Putting this on a graph we have:

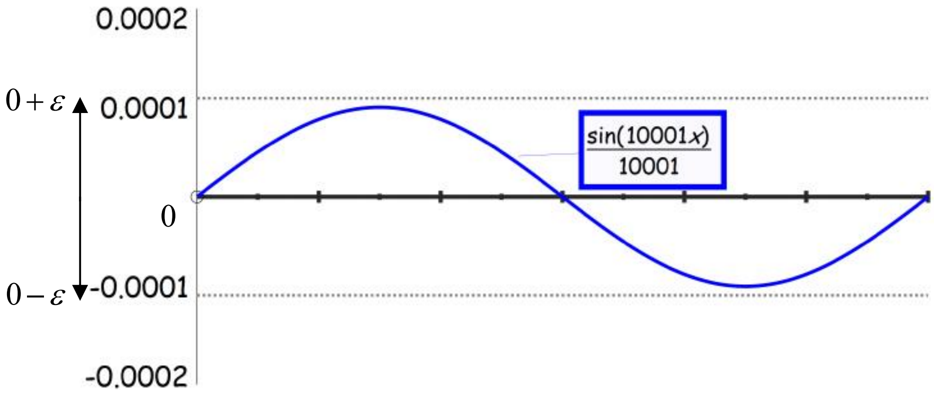

Fig 13

What value of N_0 *do we need for the given sequence of functions to be within* $\varepsilon = 0.000001$*?*

$$N_0 > \frac{1}{\varepsilon} = \frac{1}{0.000001} = 1\,000\,000$$

For $N_0 = 1\,000\,001$ and $n \geq N_0 = 1\,000\,001$ we have

$$-0.000001 < f_{1\,000\,001}(x) = \frac{\sin(1000001x)}{1000001} < 0.000001$$

Similarly the graph of this is given by:

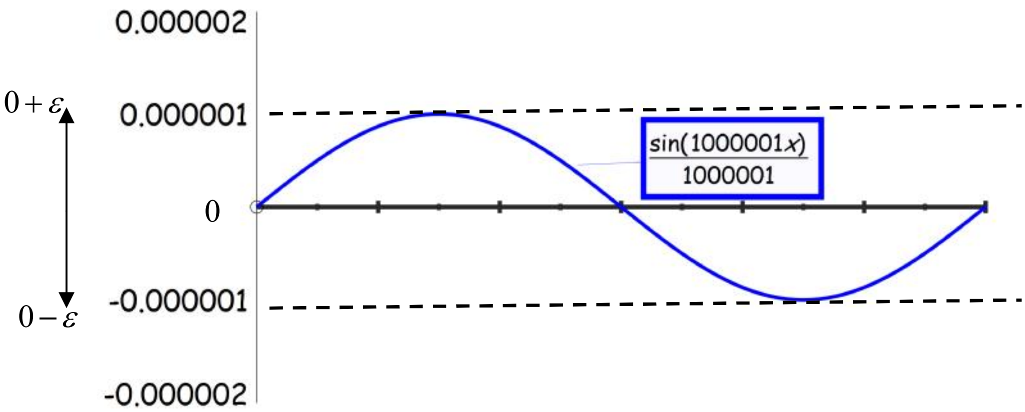

Fig 14

We need to write the formal definition of uniform convergence.

Definition (3.3). The sequence of functions $(f_n(x))$ converges **uniformly** to a function $f(x)$ in the domain *D* $\Leftrightarrow$ for every x in *D* and for every $\varepsilon > 0$ there exists a natural number N_0 (depending only on ε) such that

$$|f_n(x) - f(x)| < \varepsilon \text{ provided } n \geq N_0$$

Example 10

Let $f_n : [-2,\ 2] \to \mathbb{R}$ be a sequence of functions given by:

$$f_n(x) = \frac{x}{n}$$

Show that $f_n(x)$ converges uniformly on $[-2,\ 2]$ to the zero function $f(x) = 0$ by using the above Definition (3.3).

Solution

Our domain is $[-2,\ 2]$ which means that:

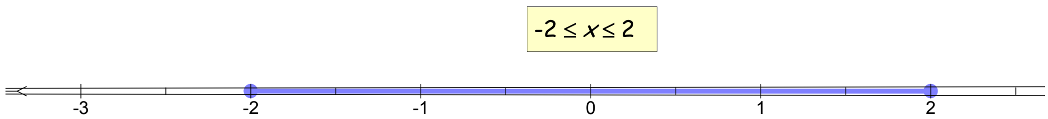

Fig 15

Let $\varepsilon > 0$ be given. Then there is a natural number N_0 such that for all $n \geq N_0$ we have

$$|f_n(x) - f(x)| = \left|\frac{x}{n} - 0\right| = \frac{|x|}{n} \leq \frac{2}{N_0} \text{ because } -2 \leq x \leq 2$$

To show that $f_n(x) = \frac{x}{n}$ converges to zero we need to find a N_0 which gives $\frac{2}{N_0} < \varepsilon$.

Transposing this gives the inequality $\frac{2}{\varepsilon} < N_0$ or writing this the other way $N_0 > \frac{2}{\varepsilon}$. Hence for all $n \geq N_0$ we have

$$|f_n(x) - 0| \leq \frac{2}{N_0} < \varepsilon \text{ provided } N_0 > \frac{2}{\varepsilon}$$

By the above Definition (3.3) we conclude that $f_n(x) = \frac{x}{n}$ converges uniformly on $[-2,\ 2]$ to the zero function $f(x) = 0$.

Note that in the above example the natural number $N_0 > \frac{2}{\varepsilon}$ depends only on ε and **not** on *x*.

Example 11

Let $f_n : [1, \quad +\infty[\to \mathbb{R}$ be a sequence of functions given by:

$$f_n(x) = e^{-nx}$$

Show that $f_n(x)$ converges uniformly on the interval $x \in [1, \quad +\infty[$ to the zero function $f(x) = 0$ by using the above Definition (3.3).

Solution

Let $\varepsilon > 0$ be given. Then there exists a natural number N_0 such that for all $n \geq N_0$ we have

$$|f_n(x) - 0| = |e^{-nx}| \underbrace{=}_{\text{Because } e^{-nx} > 0} e^{-nx} \qquad (\dagger)$$

The function e^{-nx} is a decreasing function which means that if $b \geq a > 0$ then $e^{-nb} \leq e^{-na}$. *How do we know that e^{-nx} is a decreasing function on the domain $[1, \quad +\infty[$?*

We can show the derivative of $g(x) = e^{-nx}$ is negative. Hence $g'(x) = -ne^{-nx} < 0$ because *n* is a natural number and $e^{-nx} > 0$.

Therefore with $x \in [1, \quad +\infty[$ which means that $1 \leq x < +\infty$ we have

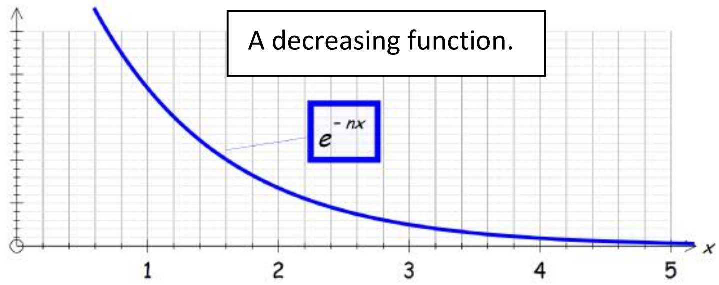

Fig 16

Since e^{-nx} is a decreasing function we have

$$e^{-nx} \leq e^{-n(1)} = e^{-n} \text{ for } x \geq 1 \qquad (*)$$

From the definition we have $n \geq N_0$ and because e^{-nx} is a decreasing function therefore

$$e^{-n} \leq e^{-N_0}$$

Note that $e^{-N_0} = \frac{1}{e^{N_0}}$. By $(\dagger)$ and (*) we have

$$\left|f_n(x)-0\right| = e^{-nx} \le e^{-n} \le \frac{1}{e^{N_0}}$$

How do we show that $f_n(x) = e^{-nx}$ *is* ***uniformly convergent*** *on* $[1, \quad +\infty[$ *?*

Required to prove that $\left|f_n(x)-0\right| = e^{-nx} \le \frac{1}{e^{N_0}} < \varepsilon$ for any $\varepsilon > 0$.

If $\varepsilon \ge 1$ then by the properties of the exponential function $\left|f_n(x)-0\right| = e^{-nx} < 1 \le \varepsilon$ for $x \in [1, \quad +\infty[$.

If $\varepsilon < 1$ then transposing the above inequality $\frac{1}{e^{N_0}} < \varepsilon$:

$$\frac{1}{e^{N_0}} < \varepsilon \quad \Leftrightarrow \quad \frac{1}{\varepsilon} < e^{N_0} \quad \Leftrightarrow \quad \ln\left(\frac{1}{\varepsilon}\right) < N_0 \text{ or the other way } N_0 > \ln\left(\frac{1}{\varepsilon}\right)$$

We choose $N_0 > \ln\left(\frac{1}{\varepsilon}\right)$. Therefore with $n \ge N_0 > \ln\left(\frac{1}{\varepsilon}\right)$ we have

$$\left|f_n(x)-0\right| = e^{-nx} \le \frac{1}{e^{N_0}} < \varepsilon$$

Note that $N_0 > \ln\left(\frac{1}{\varepsilon}\right)$ depends only on $\varepsilon > 0$ which means that $f_n(x) = e^{-nx}$ converges uniformly in the interval $[1, \quad +\infty[$.

B2 Non Uniform Convergence

Is there a relationship between ***pointwise*** *and* ***uniform*** *convergence?*

If a sequence of functions $(f_n(x))$ is uniformly convergent to $f(x)$ on an interval of the real numbers then $(f_n(x))$ is pointwise convergent to $f(x)$ on the same interval. You are asked to show this result in the next set of exercise.

It is **not** true the other way round, that is if $(f_n(x))$ converges pointwise to $f(x)$ then $(f_n(x))$ converges to $f(x)$ uniformly is false. *How do we prove a sequence of functions is* ***not*** *uniformly convergent?*

To prove that a sequence of functions is **not** uniformly convergent we negate the Definition (3.3) of uniform convergence given on page 3:

The sequence of functions $(f_n(x))$ converges **uniformly** to a function $f(x)$ in the domain *D* $\Leftrightarrow$ for every x in *D* and for every $\varepsilon > 0$ there exists a natural number N_0 (depending only on ε) such that $|f_n(x) - f(x)| < \varepsilon$ provided $n \ge N_0$.

Negating this statement we have:

Definition (3.4). The sequence of functions $(f_n(x))$ does **not** converges **uniformly** to a function $f(x)$ in the domain *D* $\Leftrightarrow$ there exists a x in *D* and an $\varepsilon > 0$ such that for every natural number *n* we have $|f_n(x) - f(x)| \ge \varepsilon$.

We can apply this definition to show a given sequence of functions is **not** uniformly convergent. We need to find a value of *x* and a value for ε such that $|f_n(x) - f(x)| \ge \varepsilon$ to show the sequence of functions $(f_n(x))$ does **not** converge uniformly to $f(x)$.

Example 12

Let $f_n : \mathbb{R} \to \mathbb{R}$ be a sequence of functions given by:

$$f_n(x) = \frac{x}{n}$$

Show that $f_n(x)$ converges pointwise for all $x \in \mathbb{R}$ to the zero function $f(x) = 0$ but **not uniformly**.

Solution

We have already shown in Example 4 on page 7 that $f_n(x) = \frac{x}{n}$ converges pointwise to the zero function.

To show that the sequence of functions $(f_n(x))$ does **not** converge uniformly to zero we need to find an $\varepsilon > 0$ and *x* such that $|f_n(x) - 0| \ge \varepsilon$.

Consider an $0 < \varepsilon < 2$ and $x = 2n$ then for all natural numbers *n* we have

$$|f_n(2n) - 0| = \left|\frac{2\not{n}}{\not{n}} - 0\right| = 2 \ge \varepsilon$$

Hence by the above definition (3.4) we conclude that $f_n(x) = \frac{x}{n}$ does **not** converge uniformly to zero for $x \in \mathbb{R}$.

Example 13

Let $f_n : \left]0, \quad +\infty\right[\to \mathbb{R}$ be a sequence of functions given by:

$$f_n(x) = \frac{1}{nx}$$

Show that $f_n(x)$ converges pointwise on the interval $\left]0, \quad +\infty\right[$ to the zero function $f(x) = 0$ but **not uniformly**.

Solution

Applying the limit theorems we have

$$\lim_{n\to\infty}\left(f_n(x)\right) = \lim_{n\to\infty}\left(\frac{1}{nx}\right) = \frac{1}{x}\left[\lim_{n\to\infty}\left(\frac{1}{n}\right)\right] = 0 \quad \left[\text{Because } \lim_{n\to\infty}\left(\frac{1}{n}\right) = 0\right]$$

Hence $f_n(x)$ converges pointwise to 0.

In order to show that the sequence of functions $\left(f_n(x)\right)$ does **not** converge uniformly to zero we need to find a ε and a value for *x* such that $\left|f_n(x) - 0\right| \geq \varepsilon$ as described in the above definition (3.4).

Let $0 < \varepsilon < 1$ and take $x = \frac{1}{n}$ then for every natural number *n* we have

$$\left|f_n\left(\frac{1}{n}\right) - f\left(\frac{1}{n}\right)\right| = \left|\frac{1}{n(1/n)} - 0\right| = \left|1 - 0\right| = 1 \geq \varepsilon$$

Since $\left|f_n(x) - 0\right| \geq \varepsilon$ we conclude that $\left(f_n(x)\right)$ does **not** converge uniformly to zero.

Example 14

Let $f_n : \mathbb{R} \to \mathbb{R}$ be a sequence of functions given by:

$$f_n(x) = \frac{x^2 + nx}{n}$$

Show that $f_n(x)$ converges pointwise for all $x \in \mathbb{R}$ to the $f(x) = x$ but **not uniformly**.

Solution

We need to first show that $\lim_{n\to\infty}\left[f_n(x)\right] = \lim_{n\to\infty}\left(\frac{x^2 + nx}{n}\right) = x$. *How?*

By applying the limit theorems. We have

$$\lim_{n\to\infty}\left(\frac{x^2+nx}{n}\right)=\lim_{n\to\infty}\left(\frac{x^2}{n}\right)+\lim_{n\to\infty}\left(\frac{\not{n}x}{\not{n}}\right)$$
$$=x^2\left[\lim_{n\to\infty}\left(\frac{1}{n}\right)\right]+\lim_{n\to\infty}(x)=x^2(0)+x=x$$

Hence the given sequence of functions $(f_n(x))$ converges pointwise to *x*.

What else do we need to show?

Required to prove that the convergence is **not** uniform. This means that we need to find a *x* in $\mathbb{R}$ and an $\varepsilon>0$ such that $|f_n(x)-x|=\left|\frac{x^2+nx}{n}-x\right|\geq\varepsilon$. Consider this distance function:

$$|f_n(n)-x|=\left|\frac{x^2+nx}{n}-x\right|=\left|\frac{x^2+nx-nx}{n}\right|=\left|\frac{x^2}{n}\right|=\frac{x^2}{n}$$

Let $x=n$ and $0<\varepsilon\leq n$ then we have

$$|f_n(x)-x|=\frac{x^2}{n}=\frac{n^2}{n}=n\geq\varepsilon$$

The given sequence of functions $(f_n(x))$ does **not** converge uniformly to *x*.

SUMMARY

The sequence of functions $(f_n(x))$ converges **uniformly** to $f(x)$ means that for all $n\geq N_0$

$$|f_n(x)-f(x)|<\varepsilon$$

where the N_0 depends on $\varepsilon>0$ and **not** on *x*.

If a sequence of functions converges uniformly to $f(x)$ then it converges pointwise. However if a sequence of functions converges pointwise then it may not converge **uniformly**. We have

$$f_n(x)\underbrace{\to}_{\text{Uniformly}}f(x)\text{ implies that }f_n(x)\underbrace{\to}_{\text{Pointwise}}f(x)$$
$$f_n(x)\underbrace{\to}_{\text{Pointwise}}f(x)\text{ does }\textbf{not}\text{ imply that }f_n(x)\underbrace{\to}_{\text{Uniformly}}f(x)$$