

SECTION B Algebra of Propositions

By the end of this section you will be able to:

- understand a truth table
- construct a truth table for compound propositions
- show equivalence of propositions using a truth table

B1 Truth Table

In this section we use a truth table to evaluate the truth value of propositions. We can make a truth table for a compound proposition by writing down the truth values of each individual proposition. Truth table is a table used to find the truth value of a compound proposition.

Let P and Q be propositions. By the definition of proposition, P can be either true or false. Similarly Q can only be either true or false. We can write this in a table as:

P	Q
True (T)	True (T)
False (F)	False (F)

TABLE 1

We will not write out the complete words true or false but use the symbols T and F respectively.

We can construct the truth table for the negation of a proposition. Let P be a proposition and $\neg P$ (not P) be the negation of P (or the opposite of P).

What is the truth value of $\neg P$ if P is true?

$\neg P$ is false

What is the truth value of $\neg P$ if P is false?

$\neg P$ is true

Hence we have the truth table of $\neg P$:

P	$\neg P$
T	F
F	T

TABLE 2

The truth table gives the truth values of $\neg P$. Reading the rows from left to right it says that if P is true then $\neg P$ is false and if P is false then $\neg P$ is true.

You need to remember the truth values of not P which has the opposite truth value to P . We use this throughout this section.

Next we construct the truth table for the connective, \wedge (and).

Let P and Q be propositions. The proposition $P \wedge Q$ (P and Q) is **only** true if **both** P is true and Q is true. Otherwise it is false.

We can compare this with everyday English. For example let P be the proposition 'it is raining' and Q be the proposition 'it is above 20°C ', then the compound proposition, 'it

is raining **and** above 20°C ' is **only** true if both 'it is raining' and 'it is above 20°C ' is true, otherwise it is false.

We can make a truth table for $P \wedge Q$ by first listing all the combination of truth values for P and Q in the first two columns.

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

TABLE 3

Again you need to remember that $P \wedge Q$ (P and Q) is **only** true if both P is true and Q is true otherwise it is false as shown in TABLE 3. We use this later in the section.

Example 12

Construct a truth table for $\neg(P \wedge Q)$.

Solution

Recall $\neg(P \wedge Q)$ means not (P and Q).

How do we find the truth value for $\neg(P \wedge Q)$?

It will be the opposite of the last column of TABLE 3 because $\neg(P \wedge Q)$ means not ($P \wedge Q$). That is if $P \wedge Q$ is true then $\neg(P \wedge Q)$ is false and vice versa. We can label the columns of the table so that we can refer to various parts of the table.

Column 1	Column 2	Column 3	Column 4
P	Q	$P \wedge Q$	$\neg(P \wedge Q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

TABLE 4

Obviously column 3 in TABLE 4 is the same as the last column in TABLE 3. The opposite of these truth values (col 3) gives the truth values in column 4.

Next we look at the truth table for $P \vee Q$.

Let p and q be propositions. The compound proposition $P \vee Q$ (P or Q) is true if either one of P and Q is true. It is **only** false when **both** P is false and Q is false. Again comparing with everyday English, let P be the proposition 'it is raining' and Q be the proposition 'it is above 20°C ', then the compound proposition, 'it is raining **or** above 20°C ' is **only** false if both 'it is raining' and 'it is above 20°C ', is false otherwise it is true.

Hence the truth table for $P \vee Q$ is given by:

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

TABLE 5

You need to remember the truth values of $P \vee Q$ (P or Q) which is **only** false if both P is false and Q is false as shown in TABLE 5.

From these three connectives \neg , \wedge and \vee we can construct more complex propositions. These three are the basis or fundamental connectives and you need to know their truth values. The remaining section relies on you knowing the truth values of these connectives.

Example 13

Construct a truth table for $\neg(P \vee Q)$.

Solution

$\neg(P \vee Q)$ means the opposite of $P \vee Q$. Hence if $P \vee Q$ is true then $\neg(P \vee Q)$ is false and vice versa. Therefore the truth value of $\neg(P \vee Q)$ is the opposite of the last column of TABLE 5.

P	Q	$P \vee Q$	$\neg(P \vee Q)$
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

TABLE 6

Note that the first three columns are the same as TABLE 5. The last column gives the truth value of $\neg(P \vee Q)$.

Example 14

Construct a truth table for $(\neg P) \wedge (\neg Q)$.

Solution

Remember the symbol \neg stands for not. $\neg P$ will have truth values opposite to p and similarly $\neg Q$ will have truth values opposite to q . *How do we find the truth values of $(\neg P) \wedge (\neg Q)$?*

Remember the connectives \wedge (and) only gives true if **both** $\neg P$ and $\neg Q$ are true else it is false. Hence we have

P	Q	$\neg P$	$\neg Q$	$(\neg P) \wedge (\neg Q)$
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

TABLE 7

B2 Equivalence

What do you notice about the truth values of $\neg(P \vee Q)$ and $(\neg P) \wedge (\neg Q)$?

Inspect the last columns of TABLE 6 and TABLE 7. Clearly $\neg(P \vee Q)$ and $(\neg P) \wedge (\neg Q)$ have the same truth values. We say the propositions $\neg(P \vee Q)$ and $(\neg P) \wedge (\neg Q)$ are **logically equivalent** or just equivalent.

In general propositions are (logically) equivalent if they have the same truth value. We can show equivalence of two compound propositions by establishing a truth table and comparing the truth value in the relevant columns. If the columns agree then the compound propositions are equivalent.

Example 15

Show that $\neg(P \wedge Q)$ and $(\neg P) \vee (\neg Q)$ are logically equivalent.

Solution

We first list the combination of truth values for P and Q in the first two columns. We now find the truth values of each part. Hence we have:

Column 1	Column 2	Column 3	Column 4	Column 5	Column 6	Column 7
P	Q	$P \wedge Q$	$\neg(P \wedge Q)$	$\neg P$	$\neg Q$	$(\neg P) \vee (\neg Q)$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

TABLE 9

Note that $\neg(P \wedge Q)$ (column 4) and $(\neg P) \vee (\neg Q)$ (column 7) are the same. Hence $\neg(P \wedge Q)$ and $(\neg P) \vee (\neg Q)$ are equivalent. This is normally written as

$$\neg(P \wedge Q) \equiv (\neg P) \vee (\neg Q)$$

The equivalent symbol is \equiv .

The next example might seem laborious compared to the above examples because you need to be careful in listing all the possible combinations of the truth values for P , Q and R . There is also cause for confusion because it is easy to make a mistake between the various columns. Therefore we have numbered each column. Be careful in carrying out the correct operation among these columns.

Example 16

Show that $P \vee (Q \wedge R)$ and $(P \vee Q) \wedge (P \vee R)$ are logically equivalent.

Solution

We have to construct a truth table for these propositions.

For the three propositions P , Q and R we need 8 rows covering **all** the possible truth values as shown in the first three columns of TABLE 10.

Column 1	Column 2	Column 3	Column 4	Column 5	Column 6	Column 7	Column 8
P	Q	R	$Q \wedge R$	$P \vee (Q \wedge R)$	$P \vee Q$	$P \vee R$	$(P \vee Q) \wedge (P \vee R)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
F	T	T	T	T	T	T	T
T	F	F	F	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

TABLE 10

You might have difficulty in following this table but the references to the columns and how they are connected might help.

The first three columns list the combination of P , Q and R .

$$\text{Column 4} = (\text{Column 2}) \wedge (\text{Column 3})$$

$$\text{Column 5} = (\text{Column 1}) \vee (\text{Column 4})$$

$$\text{Column 6} = (\text{Column 1}) \vee (\text{Column 2})$$

$$\text{Column 7} = (\text{Column 1}) \vee (\text{Column 3})$$

$$\text{Column 8} = (\text{Column 6}) \wedge (\text{Column 7})$$

Since column 5 has the same truth value as column 8 therefore

$$P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R) \quad [\text{Equivalent}]$$

SUMMARY

We have constructed truth tables for the various compound propositions. We have also used truth tables to establish equivalence of propositions.