Chapter 5 Real Numbers

Section A **Upper and Lower Bounds**

By the end of this section you will be able to
- understand what is meant by upper and lower bound of a set
- understand what is meant by Least Upper Bound (LUB) or supremum and Greatest Lower bound (GLB) or infimum
- determine infimum and supremum of a set

In this section we examine non-empty subsets of the real numbers, $\mathbb{R}$. We first look at bounded sets and then define the supremum and infimum of a set.

A1 **Bounded Sets**

We define what is meant by upper and lower bounds of a set.

Definition (5.1).
Let S be a non-empty subset of the real numbers $\mathbb{R}$.
a) We say the set S is **bounded above** if there is a real number M such that

$$x \le M \qquad \text{For All } x \text{ in } S$$

The real number M is called an **upper bound** of the set S.

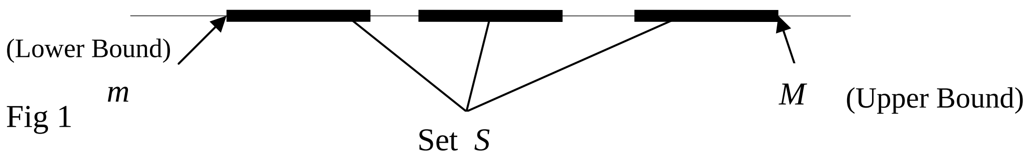


Fig 1

b) We say the set S is bounded below if there is a real number m such that

$$x \ge m \qquad \text{For All } x \text{ in } S$$

This real number m is called a **lower bound** of the set S.
c) If a set S is **both** bounded above and below then we say the set S is **bounded**. The set in Fig 1 is bounded. If the set S is **not** bounded above or below then we say the set S is **unbounded**.

A2 **Upper Bounds**

Example 1
Let $S = \{x \mid x < 1 \text{ and } x \in \mathbb{R}\}$
a) Is S bounded above? If it is, find an upper bound.
b) Is S bounded below? If it is, determine a lower bound.
c) Is S bounded or unbounded?

Solution.
What does $S = \{x \mid x < 1 \text{ and } x \in \mathbb{R}\}$ *mean?*
It is the set of all the real numbers less than 1.

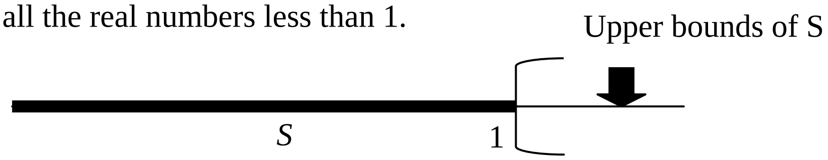


Fig 2

(a) The set $S = \{x \mid x < 1 \text{ and } x \in \mathbb{R}\}$ is bounded above by 1 because for all $x \in S$ we have

$$x < 1$$

All the elements in the set are less than 1. Hence 1 is an upper bound of S. Any real number greater than 1 is also an upper bound of S such as 2, 3 , 4 , 666, 10 000 etc

(b) The set S is **not** bounded below because there is no real number m such that for all $x \in S$

$$x \geq m$$

See Figure 2. There is **no** lower bound.

(c) The set S is **unbounded** because it is **not** bounded below even though it is bounded above.

You might have found the answer to part (a) in Example 1 unusual because you were asked to find an upper bound of the set S and there were an infinite number of them, any number greater than or equal to 1 will do. *However we prefer just to have one upper bound of the set S but which one?*

The least of these upper bounds. *What is the value of the Least Upper Bound of the set S ?*

1 because all the other upper bounds are greater than 1. The Least Upper Bound denoted by **LUB** is an important upper bound called the **supremum** of S.

Fig 3

Next we give the definition of LUB or supremum.

Definition (5.2)

Let S be a non-empty subset of $\mathbb{R}$ which is bounded above. Then a real number u is the supremum of the set S if

(a) u is an upper bound of the set S

(b) u' is any other upper bound of the set S then $u \leq u'$

This means that the supremum is the real number which is the Least Upper Bound of a set S and it is denoted by sup(S) and is pronounced ‘soup S’. In definition (5.2) $u = \sup(S)$.

For example let $S = \{1,\ 2,\ 3\}$ then $\sup(S) = 3$.

Example 2

(a) Let $A = \{x \mid x < 2 \text{ and } x \in \mathbb{R}\}$

(b) Let $B = \{\cos(x) \mid x \in \mathbb{R}\}$

(c) Let $C = \left\{\frac{1}{n} \,\middle|\, n \in \mathbb{N}\right\}$

(d) Let $D = \left\{\frac{1}{1+n^2} \,\middle|\, n \in \mathbb{N}\right\}$

Determine the real numbers $\sup(A)$, $\sup(B)$, $\sup(C)$ and $\sup(D)$.

Solution.

(a) *What does* $A = \{x \mid x < 2 \text{ and } x \in \mathbb{R}\}$ *mean?*

It is all the real numbers up to 2, that is

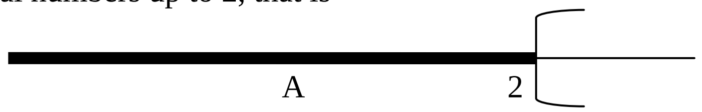


Fig 4

What is the supremum of A*?*

It is the Least Upper Bound, LUB, of A. Since 2 is an upper bound of A and it is the smallest upper bound therefore $\sup(A) = 2$. Note that $2 \notin A$ that is 2 is not a member of the set A. The supremum need **not** be a member of the set.

(b) *What does* $B = \{\cos(x) \mid x \in \mathbb{R}\}$ *mean?*

It is the values of $\cos(x)$ for all real values of x. *What does* $\cos(x)$ *lie between for all real values of* x*?*

Remember from your previous studies in trigonometry that $-1 \le \cos(x) \le +1$.

Therefore B is the set of all real numbers between -1 to 1.

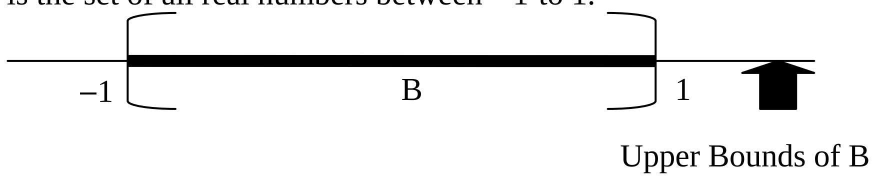


Fig 5

What is the value of an upper bound of the set B*?*

There are an infinite number of upper bounds of the set B such as 1, 1.5, 3.7, 42, 4 000 002 etc *But what is the Least Upper Bound, LUB, of the set* B*?*

1 is the LUB because it is smallest of all the upper bounds. Remember the LUB is the supremum, therefore $\sup(B) = 1$.

(c) *What does* $C = \left\{\frac{1}{n} \,\middle|\, n \in \mathbb{N}\right\}$ *mean?*

It is the set of real numbers equal to $\frac{1}{n}$ for the natural numbers $n = 1, 2, 3, 4, 5$ etc

$$C = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\right\}$$

Is the set C *bounded above?*

Yes it is bounded above because for all $n \in \mathbb{N}$

$$\frac{1}{n} \le 1$$

Is 1 the Least Upper Bound, LUB, of the set C?

Yes because it is least of all the other upper bounds of C. *What is the supremum of C, sup*(C)*, equal to?*

LUB of C is 1, hence $\sup(C) = 1$. Remember the Least Upper Bound is the supremum.

(d) *What does* $D = \left\{\frac{1}{1+n^2} \,\middle|\, n \in \mathbb{N}\right\}$ *mean?*

D is the set of real numbers $\frac{1}{1+n^2}$ for $n = 1, 2, 3, 4, 5, \cdots$

$$D=\left\{\frac{1}{1+1^2},\ \frac{1}{1+2^2},\ \frac{1}{1+3^2},\ \frac{1}{1+4^2},\ \frac{1}{1+5^2}\cdots\right\}=\left\{\frac{1}{2},\ \frac{1}{5},\ \frac{1}{10},\ \frac{1}{17},\ \frac{1}{26}\cdots\right\}$$

Does the set D have an upper bound, if it does then what is the value of this upper bound?

Upper bounds of D are $\frac{1}{2}$, 1, e, π, 1 000 etc. Remember upper bounds do **not** need to be members of the set. *Which one is the Least Upper Bound, LUB?*

Least Upper Bound of the set D is $\frac{1}{2}$, hence $\sup(D)=\frac{1}{2}$.

A3 **Lower Bounds**

Example 3

Let $T=\{x \mid x>-2 \text{ and } x\in\mathbb{R}\}$

a) Is T bounded above? If it is, find an upper bound.
b) Is T bounded below? If it is, determine a lower bound.

Solution.

What does $T=\{x \mid x>-2 \text{ and } x\in\mathbb{R}\}$ *mean?*

It is the set of all the real numbers greater than -2.

Fig 6

a) The given set $T=\{x \mid x>-2 \text{ and } x\in\mathbb{R}\}$ is **not** bounded above because there is **no** real number, M, such that

$$x\le M \qquad \text{For All } x\in T$$

Since the set is **not** bounded above therefore the set T does **not** have an upper bound.

b) The set $T=\{x \mid x>-2 \text{ and } x\in\mathbb{R}\}$ is bounded below because all members of this set are greater than -2 by the definition of the set T.

A lower bound of T is -2 but any number less than -2 is also a lower bound of T such as $-3, -4, -4.78, -11, -1\,000, \cdots$ There are an infinite number of lower bounds of T. Note that -2 is **not** a member of the set T because $T=\{x \mid x>-2 \text{ and } x\in\mathbb{R}\}$.

We have **not** really answered part (b) of Example 3 because we were asked to find a lower bound of the set T but we found an infinite number of them and there was no unique answer. *So which one should we choose as our lower bound of T ?*

Greatest Lower Bound which is -2. The Greatest Lower Bound denoted GLB is an important lower bound which is called the **infimum** of the set T.

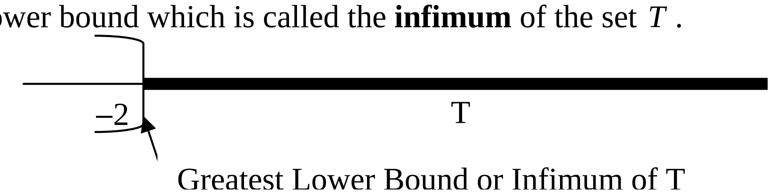


Fig 7

The general definition of the infimum or Greatest Lower Bound, GLB, is given next.

Definition (5.3)
Let S be a non-empty subset of $\mathbb{R}$ which is bounded below. Then a real number l is the **infimum** of S if

a) l is a lower bound of the set S

b) l' is any other lower bound of the set S then $l' \le l$

This means that the infimum is the real number which is the Greatest Lower Bound, GLB, of the set S and it is denoted by $\inf(S)$.

For example let $S = \{1,\ 2,\ 3\}$ then $\inf(S) = 1$.

Example 4

(a) Let $A = \{x \mid x > 99 \text{ and } x \in \mathbb{R}\}$

(b) Let $B = \{\sin(x) \mid x \in \mathbb{R}\}$

(c) Let $C = \left\{ \frac{1}{n} \,\middle|\, n \in \mathbb{N} \right\}$

(d) Let $D = \{1 - (-1)^n \mid n \in \mathbb{N}\}$

Find the real numbers $\inf(A)$, $\inf(B)$, $\inf(C)$ and $\inf(D)$.

Solution.

(a) *What does* $A = \{x \mid x > 99 \text{ and } x \in \mathbb{R}\}$ *mean?*

It is the set of all real numbers greater than 99:

Lower Bounds of A

Fig 8

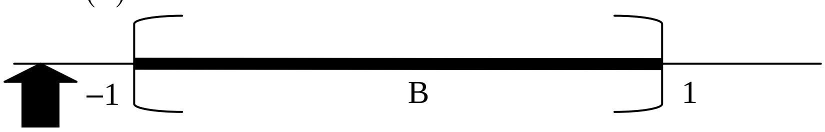
What is the Greatest Lower Bound of A*?*

99 because 99 is a lower bound and it is the largest lower bound. All the other lower bounds of A such as 98, 90, π, 1 etc are less than 99. *So what is the infimum of* A*?* It is the Greatest Lower Bound, GLB, of A which is 99. Hence $\inf(A) = 99$. Note that 99 is **not** a member of the set A because it is the set of all real numbers greater than 99, $A = \{x \mid x > 99 \text{ and } x \in \mathbb{R}\}$.

(b) *What does* $B = \{\sin(x) \mid x \in \mathbb{R}\}$ *mean?*

This is the set of numbers equal to $\sin(x)$ where x is a real number. *What are the values of* $\sin(x)$ *for real* x*?*

Remember from your previous studies in trigonometry that $\sin(x)$ lies between -1 to $+1$, that is $-1 \le \sin(x) \le +1$. Hence B is the set of real numbers between -1 to 1:



Lower Bounds of B

Fig 9

What is a lower bound of the set B*?*

-1 is a lower bound but so is $-2, -3, -4, \cdots$ *What is the Greatest Lower Bound, GLB, of the set B?*
-1 because all the other lower bounds are less than -1. Remember the Greatest Lower Bound is the infimum therefore $\inf(B) = -1$.

(c) This is the same set C as in Example 2:

$$C = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\right\}$$

Is the set C bounded below?
Yes because it does not have negative values. So all the negative numbers are lower bounds of C. *What is the Greatest Lower Bound, GLB, of the set C?*
It is 0 because as n gets larger the size of $\frac{1}{n}$ gets smaller but is never negative. Hence the infimum is the GLB so we have $\inf(C) = 0$. Note that 0 is **not** a member of the set C.

(d) *What does* $D = \left\{1-(-1)^n \,\middle|\, n \in \mathbb{N}\right\}$ *represent?*

$$D = \left\{1-(-1),\quad 1-(-1)^2,\quad 1-(-1)^3,\quad 1-(-1)^4, \cdots\right\}$$
$$= \{2,\quad 0,\quad 2,\quad 0, \cdots\}$$
$$= \{0,\quad 2\}$$

Is the set D bounded below?
Yes by 0 and all the negative real numbers. *What is the Greatest Lower Bound, GLB, of the set D equal to?*
0. Hence the infimum of D is 0 that is $\inf(D) = 0$. Note that 0 is a member of the set D.

A4 **Examples of Sets**

Example 5
Consider the set

$$A = \left\{1 - \frac{1}{n} \,\middle|\, n \in \mathbb{N}\right\}$$

Determine $\sup(A)$ and $\inf(A)$. *Are* $\sup(A)$ *and* $\inf(A)$ *members of the set* A*?*
Solution.
What does $A = \left\{1 - \frac{1}{n} \,\middle|\, n \in \mathbb{N}\right\}$ *mean?*
It is the set of real numbers equal to $1 - \frac{1}{n}$ for the natural numbers $n = 1, \quad 2, \quad 3, \quad 4, \quad 5, \cdots$

$$A = \left\{1-\frac{1}{1},\quad 1-\frac{1}{2},\quad 1-\frac{1}{3},\quad 1-\frac{1}{4},\quad 1-\frac{1}{5}, \cdots\right\}$$
$$= \left\{0,\quad \frac{1}{2},\quad \frac{2}{3},\quad \frac{3}{4},\quad \frac{4}{5}, \cdots\right\}$$

Does the set A have an upper bound?

For all $n \in \mathbb{N}$, $1-\frac{1}{n} \in A$. *What is the value of* $1-\frac{1}{n}$ *for large* n*?*

As n gets larger $\frac{1}{n}$ gets smaller so $1-\frac{1}{n}$ gets closer to 1. *Is 1 an upper bound of the set A?*

Yes because for all $n \in \mathbb{N}$, $1-\frac{1}{n}$ never exceeds 1. *But is 1 the Least Upper Bound, LUB, of the set A?*

Yes because all the other upper bounds such as 1.007, 2, e, π etc are all greater than 1. Hence the LUB of the set A is 1 therefore $\sup(A)=1$. But 1 is not a member of the set A, therefore $\sup(A) \notin A$.

Now we examine the infimum that is the Greatest Lower Bound. *Is the set* A *bounded below?*

Yes because the set A contains **no** negative elements therefore all the negative numbers and 0 are lower bounds of the set A. *But what is the Greatest Lower Bound, GLB, of the set* $A=\left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \cdots\right\}$ *?*

Clearly the Greatest Lower Bound is 0 because all the negative numbers are less than 0. Since the GLB is the infimum therefore $\inf(A)=0$. *Is 0 a member of the set A?*

Yes $0 \in A$ so we have $\inf(A) \in A$.

Summarizing the above we have $\sup(A)=1$ and $\inf(A)=0$. Also $\sup(A) \notin A$ and $\inf(A) \in A$.

Example 6

Give an example of a set B with $\inf(B)=e$ but $e \notin B$.

Solution.

What is the set B *going to look like?*

The set B needs to be bounded below with the Greatest Lower Bound equal to e but e is **not** a member of the set B. Consider the set $B=\{x \mid x>e \text{ and } x \in \mathbb{R}\}$.

Fig 10

Example 7

Give an example of a set, C, which is neither bounded above or below. *What is* $\sup(C)$ *and* $\inf(C)$ *equal to?*

Solution.

Consider the set $C=\mathbb{Z}$ that is the set of all integers.

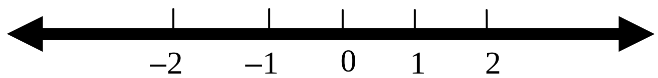


Fig 11

Clearly the set C is **not** bounded above because there is **no** real number M such that for all x in C, that is all x in $\mathbb{Z}$, we have

$$x \le M$$

Similarly C is not bounded below.

Since the set C is not bounded above therefore it **cannot** have a supremum because the definition, (5.2), of supremum says that the set needs to be bounded above to have a supremum. The supremum of C does **not** exist.

Similarly the set C is **not** bounded below therefore infimum of C does **not** exist.

SUMMARY

Let S be a non-empty subset of real numbers.

We say the set S is **bounded above** if there is a real number M such that

$$x \le M \qquad \text{For All } x \text{ in } S$$

The real number M is called an **upper bound** of the set S.

We say the set S is bounded below if there is a real number m such that

$$x \ge m \qquad \text{For All } x \text{ in } S$$

This real number m is called a **lower bound** of the set S.

Definition (5.2)

Let S be a non-empty subset of $\mathbb{R}$ which is bounded above. Then a real number u is the supremum of the set S if

(a) u is an upper bound of the set S

(b) u' is any other upper bound of the set S then $u \le u'$

The **supremum** is the **Least Upper Bound, LUB,** of the set.

Definition (5.3)

Let S be a non-empty subset of $\mathbb{R}$ which is bounded below. Then a real number l is the infimum of S if

a) l is a lower bound of the set S

b) l' is any other lower bound of the set S then $l' \le l$

The **infimum** is the **Greatest Lower Bound, GLB,** of the set.