Chapter 2 : Infinite Series

Section G **Alternate Series**

By the end of this section you will be able to

- understand what is meant by alternating series
- proof the alternating series test
- test an alternating series for convergence
- establish whether a function is increasing or decreasing
- show that a given alternating series converges
- prove properties of a general alternating series

G1 **Alternate Series**

Until now we have **only** investigated infinite series $\sum(a_n)$ where a_n have **all** been positive or zero, that is for all $n \in \mathbb{N}$, $a_n \geq 0$.

In this section we examine the infinite series $\sum(a_n)$ where some of the terms may be negative. For example

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots \qquad (*)$$

is an alternating series because positive and negative terms alternate between each other. *How do we write this series (*) in compact $\sum$ format?*

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots = \sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{1}{n}\right)$$

Can you think of any other examples of alternating series?

The following are **all** alternating series:

$$\frac{1}{4} - \frac{1}{9} + \frac{1}{16} - \frac{1}{25} + \frac{1}{36} - \ldots = \sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{1}{n^2}\right)$$

$$-1 + \frac{1}{4} - \frac{1}{9} + \frac{1}{16} - \frac{1}{25} + \frac{1}{36} - \ldots = \sum_{n=1}^{\infty} (-1)^{n} \left(\frac{1}{n^2}\right)$$

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \ldots = \sum_{n=0}^{\infty} \left(\frac{-1}{2}\right)^n$$

However the following is **not** an alternating series:

$$1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} - \frac{1}{7} + \ldots$$

because we have two negative terms and then two positive terms.

A series of the form $\sum\left((-1)^{n+1} a_n\right)$ where $a_n \in \mathbb{R}^+$ is an **alternating series**. The $(-1)^{n+1}$ gives positive or negative sign of the term a_n.

In the above examples a_n is $1/n$, $1/n^2$ and $1/2^n$ respectively.

We need to establish a test for alternating series because all previous tests **cannot** be applied to these series.Testing an alternating series is challenging because it involves checking three conditions given below. The first condition is straightforward and can be tested by inspection of the series. However checking the other two conditions involves a lot more work because we need to use techniques such as evaluating a limit and establishing a general inequality.

Alternating Series Test (2.17).
Let $\sum\left((-1)^{n+1} a_n\right)$ be a series and a_n satisfy the following:

1) $a_n > 0$ for all $n \in \mathbb{N}$
2) $\lim\limits_{n\to\infty}(a_n) = 0$
3) $a_{n+1} < a_n$ for all $n \in \mathbb{N}$

then the alternating series $\sum\left((-1)^{n+1} a_n\right)$ converges.

Note: *What does the alternating series test say?*
It says that if a_n of the general term $(-1)^{n+1} a_n$ satisfy **all three** conditions above then the alternating series $\sum\left((-1)^{n+1} a_n\right)$ converges. *What do the three conditions mean?*

1) Means $a_1,\ a_2,\ a_3,\ a_4,\ a_5,\ \ldots, a_n,\ \ldots$ are **all positive**.
2) Means the nth term converges to zero.
3) Means (n+1)th term, a_{n+1}, is less than the nth term, a_n. That is

$$a_2 < a_1,\ \ a_3 < a_2,\ \ a_4 < a_3,\ \ a_5 < a_4,\ \ a_6 < a_5, \ldots$$

the present term is smaller than the previous term in the series. It says that (a_n) is a **decreasing** sequence. The a_n is getting smaller as n is getting larger.

Notice that we examine conditions on the term a_n and **not** on $(-1)^{n+1} a_n$. *Are there any limitations of the test?*
Yes this test will **only** tell us whether a given series converges and **not** when it diverges. Also if any of the conditions are **not** satisfied then we **cannot** conclude whether the series converges or diverges and we have to apply another test.
Proof.
Consider a general alternating series which satisfies the given three conditions:

$$\sum_{n=1}^{\infty}\left((-1)^{n+1} a_n\right) = a_1 - a_2 + a_3 - a_4 + a_5 - \ldots$$

What do we need to prove?
We need to show this alternating series converges. *How?*
By considering partial sums. Let S_K be the Kth partial sum

$$S_K = a_1 - a_2 + a_3 - a_4 + a_5 - \ldots\ a_K$$

What is 2Kth partial sum, S_{2K}, equal to?

$$S_{2K} = a_1 - a_2 + a_3 - a_4 + a_5 - \ldots\ - a_{2K-2} + a_{2K-1} - a_{2K} \qquad (\dagger)$$

This can be rewritten as

$$S_{2K} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \ldots\ - (a_{2K-2} - a_{2K-1}) - a_{2K} \qquad (\dagger\dagger)$$

All the bracketed terms, $(a_2 - a_3)$, $(a_4 - a_5)$, etc are positive. *Why?*
Because by condition 3) we have a decreasing sequence

$$a_3 < a_2,\ a_5 < a_4,\ \ldots,\ a_{2K-1} < a_{2K-2}$$

and all the a_n's are positive by condition 1). Therefore

$$S_{2K} < a_1 \qquad \text{[2Kth partial sum is less than the first term } a_1\text{]}$$

Why?

Because S_{2K} is equal to a_1 minus positive terms (see (††)). Hence S_{2K} is bounded.

Also (S_{2K}) is an increasing sequence because by rewriting (†) we have

$$S_{2K} = (a_1 - a_2) + (a_3 - a_4) + (a_5 - a_6) + \dots + (a_{2K-1} - a_{2K})$$

and each bracketed term is positive. *Why?*

Because by condition 3) we have $a_{n+1} < a_n$ so therefore

$$a_2 < a_1,\ a_4 < a_3,\ a_6 < a_5,\ \dots,\ a_{2K} < a_{2K-1}$$

Hence (S_{2K}) is a bounded monotonic (increasing) sequence therefore by the monotonic sequence theorem, (1.??), the sequence (S_{2K}) converges. Let $L = \lim_{K\to\infty}(S_{2K})$. *But what about if the sum ends with an odd subscript such as* $2K+1$*?*

Consider S_{2K+1}. *What is the* $(2K+1)$*th partial sum,* S_{2K+1}*, equal to?*

$$S_{2K+1} = \underbrace{a_1 - a_2 + a_3 - a_4 + a_5 - \dots + a_{2K-1} - a_{2K}}_{=S_{2K} \text{ by } (\dagger)} + a_{2K+1}$$

$$= S_{2K} + a_{2K+1}$$

How do we work out the limiting value of this?

$$\lim_{K\to\infty}(S_{2K+1}) = \lim_{K\to\infty}(S_{2K} + a_{2K+1}) \qquad \text{[From Above]}$$

$$= \underbrace{\lim_{K\to\infty}(S_{2K})}_{=L \text{ (From Above)}} + \underbrace{\lim_{K\to\infty}(a_{2K+1})}_{=0} = L$$

Why is $\lim_{K\to\infty}(a_{2K+1}) = 0$*?*

Because condition 2) says $\lim_{n\to\infty}(a_n) = 0$. Since $\lim_{K\to\infty}(S_{2K}) = \lim_{K\to\infty}(S_{2K+1}) = L$ therefore by proposition (1.??) we have $\lim_{n\to\infty}(S_n) = L$ that is

$$\sum\left((-1)^{n+1} a_n\right) = a_1 - a_2 + a_3 - a_4 + a_5 - \dots = L$$

We have proven that if the alternating series, $\sum\left((-1)^{n+1} a_n\right)$, satisfies the given three conditions then it converges.

■

G2 **Testing Alternate Series**

Example 37

Test the following series for convergence:

$$\sum_{n=1}^{\infty}\left(\frac{(-1)^{n+1}}{4n-1}\right)$$

Solution

What type of series do we have?

Clearly it is an alternating series because

(1.??) A bounded monotonic sequence converges.

(1.??) If $\lim_{K\to\infty}\left(x_{2K+1}\right)=\lim_{K\to\infty}\left(x_{2K}\right)=L$ then $\lim_{n\to\infty}\left(x_n\right)=L$

$$\sum_{n=1}^{\infty}\left(\frac{(-1)^{n+1}}{4n-1}\right)=\underbrace{\frac{(-1)^2}{(4\times 1)-1}}_{n=1}+\underbrace{\frac{(-1)^3}{(4\times 2)-1}}_{n=2}+\underbrace{\frac{(-1)^4}{(4\times 3)-1}}_{n=3}+\underbrace{\frac{(-1)^5}{(4\times 4)-1}}_{n=4}\cdots$$

$$=\frac{1}{3}-\frac{1}{7}+\frac{1}{11}-\frac{1}{15}\cdots+\frac{(-1)^{n+1}}{4n-1}\cdots \qquad (\dagger)$$

How do we test this series?

By applying the alternating series test, (2.17). Let $a_n=\frac{1}{4n-1}$ then we need to check the three conditions of the test for the given series to converge.

1) Well $a_n=\frac{1}{4n-1}>0$ [Positive] for all $n\in\mathbb{N}$

2) Need to check $\lim_{n\to\infty}\left(a_n\right)=0$. Substituting $a_n=\frac{1}{4n-1}$ gives

$$\lim_{n\to\infty}\left(\frac{1}{4n-1}\right)=0$$

Thus condition 2) is also satisfied.

3) Need to check that the a_n terms are decreasing, that is $a_{n+1}<a_n$. *Is it enough to look at (†) and say* $\ldots\frac{1}{15}<\frac{1}{11}<\frac{1}{7}<\frac{1}{3}$ *therefore we have a decreasing sequence?*

No we have to show the general inequality $a_{n+1}<a_n$.

First we determine a_{n+1} by substituting $n+1$ for n in $a_n=\frac{1}{4n-1}$:

$$a_{n+1}=\frac{1}{4(n+1)-1}=\frac{1}{4n+3}$$

$$4n+3>4n-1 \qquad \left[\text{Comparing Denominators of } a_{n+1} \text{ and } a_n\right]$$

$$\frac{1}{4n+3}<\frac{1}{4n-1} \qquad \left[\text{Because } x>y \text{ implies } \frac{1}{x}<\frac{1}{y}\right]$$

Hence $a_{n+1}=\frac{1}{4n+3}<\frac{1}{4n-1}=a_n$ that is $a_{n+1}<a_n$ is satisfied.

Since all three conditions are satisfied therefore by the alternating series test (2.17) the given series $\sum_{n=1}^{\infty}\left(\frac{(-1)^{n+1}}{4n-1}\right)$ converges.

Example 38

Discuss the convergence or divergence of the following series:

$$\sum\left(\frac{(-1)^{n+1}n}{2n+1}\right)$$

Solution.

Do we have an alternating series?

Yes because the general term contains $(-1)^{n+1}$ which alters between positive and

negative signs for each n. Let $a_n = \dfrac{n}{2n+1}$ then a_n is positive for all $n \in \mathbb{N}$ so condition 1) of the alternating series test is satisfied. *What do we do next?*
Check condition 2) that is $\lim\limits_{n\to\infty}(a_n) = 0$. *How?*

Substituting $a_n = \dfrac{n}{2n+1}$ we have

$$\begin{aligned}\lim_{n\to\infty}(a_n) &= \lim_{n\to\infty}\left(\frac{n}{2n+1}\right)\\ &= \lim_{n\to\infty}\left(\frac{1}{2+1/n}\right) \qquad \begin{bmatrix}\text{Dividing Numerator}\\ \text{and Denominator by } n\end{bmatrix}\\ &= \frac{1}{2+\lim\limits_{n\to\infty}(1/n)} = \frac{1}{2+0} = \frac{1}{2} \neq 0 \qquad [\text{Not Zero}]\end{aligned}$$

Since condition 2) is **not** satisfied therefore there is no point checking condition 3).
We need to apply another test to check for convergence but which one?
The given series might diverge therefore it is worth checking that the nth term does not converge to zero.

$$\lim_{n\to\infty}\left(\frac{(-1)^{n+1}n}{2n+1}\right) = \lim_{n\to\infty}\left(\frac{(-1)^{n+1}}{2+1/n}\right)$$

This limit does **not** exist because of $(-1)^{n+1}$ therefore **cannot** equal zero. Hence by (2.6) the given series $\sum\left(\dfrac{(-1)^{n+1}n}{2n+1}\right)$ diverges.

G3 **Increasing and Decreasing Theorem**

The difficulty with using the alternating series test is to show condition 3) that is the decreasing inequality $a_{n+1} < a_n$. It is not an easy task to show this general inequality. However we can use a theorem from differentiation for this purpose. The following theorem and the rest of this subsection is a digression from infinite series but we need it so that we can apply the alternating series test to a wide range of series.

Increasing and Decreasing Theorem (2.18).
Let $a,\ b$ be real numbers with $a < b$. Let $f : [a,b] \to \mathbb{R}$ be a function which is differentiable on $[a,b]$. Then

(I) $f'(x) > 0$ for all $x \in]a,b[\quad \Rightarrow \quad f$ is **increasing** on $[a,b]$

(II) $f'(x) < 0$ for all $x \in]a,b[\quad \Rightarrow \quad f$ is **decreasing** on $[a,b]$

Proof. Omitted.

What does theorem (2.18) mean?
Part (I) means that if the derivative of f is **positive** in the interval $]a,b[$ then f is **increasing** in $[a,b]$. *What does f is increasing mean?*
We say a function f is (strictly) increasing if

$$f(y) > f(x) \quad \text{whenever} \quad y > x$$

(2.6) If $\lim\limits_{n\to\infty}(a_n) \neq 0$ then $\sum(a_n)$ diverges

To be pedantic the theorem should say **strictly** increasing and strictly decreasing. In this section when we are referring to increasing or decreasing we mean strictly increasing and strictly decreasing respectively. *What is the difference?*

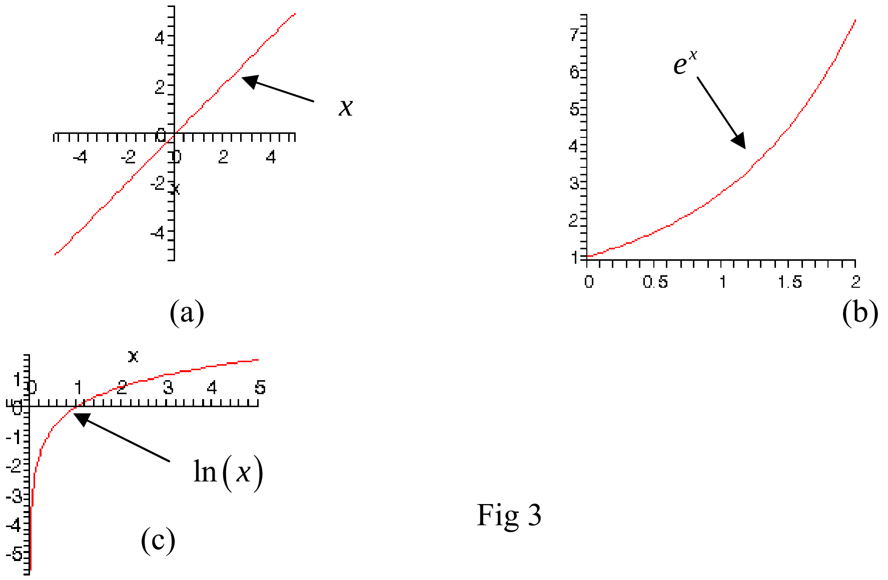
$$\text{Strictly Increasing } f(y) > f(x) \text{ whenever } y > x$$

$$\text{Increasing } f(y) \geq f(x) \text{ whenever } y > x$$

That is for strictly increasing, $f(y)$ is strictly greater than $f(x)$, whilst for increasing, $f(y)$ is greater than or equal to $f(x)$ whenever $y > x$.

In general if we have a **positive** derivative then the function is **increasing**.

Examples of increasing functions are x, e^x and $\ln(x)$ as you can observe in the following graphs:



(a) (b) (c)

Fig 3

What does part (II) of theorem (2.18) mean?

Part (II) means that if the derivative of f is negative in the interval $]a,b[$ then f is **decreasing** in $[a,b]$. *What does decreasing mean?*

We say a function f is decreasing if

$$f(y) < f(x) \text{ whenever } y > x$$

Remember we are talking about strictly decreasing functions.

In general if the derivative is **negative** then the function is **decreasing**.

Examples of decreasing functions are $-x$, e^{-x} and $1/x$ $(x \neq 0)$ as you can observe in the following graphs:

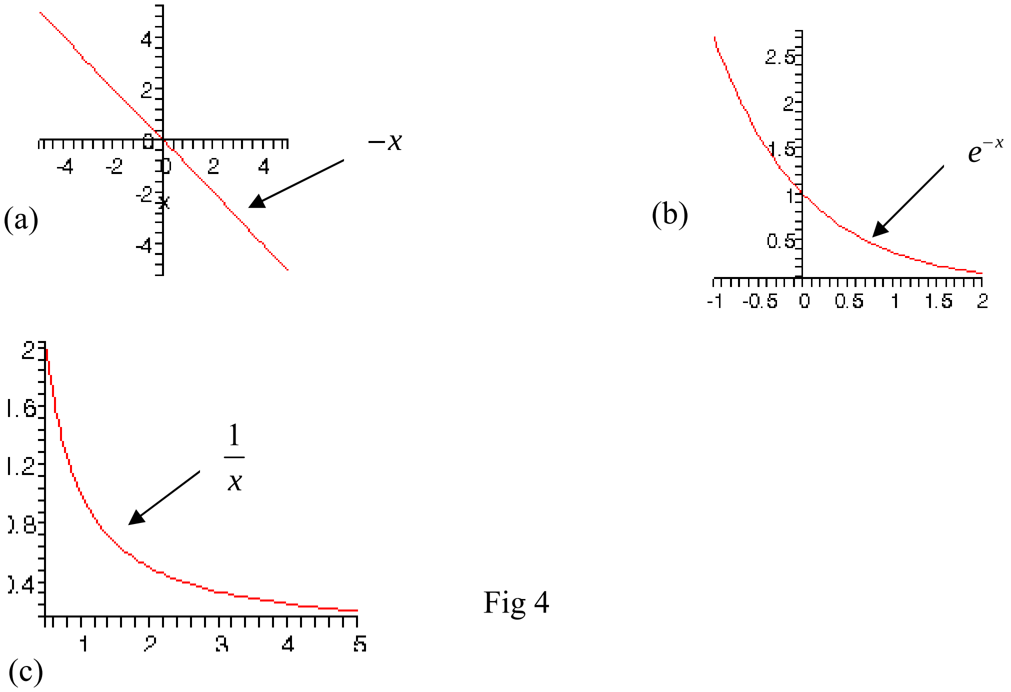


Fig 4

Next we use the theorem to establish whether the given functions are increasing or decreasing.

Example 38

Show that logarithmic function, $\ln(x)$, is increasing for $x>0$.

Solution.

Let $f(x)=\ln(x)$ for $x>0$. Then

$$f'(x)=\frac{1}{x} \qquad \left[\begin{array}{l}\text{Because Differentiating } \ln(x) \\ \text{gives } 1/x\end{array}\right]$$

Since $f'(x)=\frac{1}{x}>0$ for $x>0$ therefore by theorem (2.18) part (I) the function $\ln(x)$ is increasing for $x>0$. Remember positive derivative gives an increasing function.

Example 39

Show that $\sin(x)$ is increasing in the interval $\left[0,\frac{\pi}{2}\right]$.

Solution.

Let $f(x)=\sin(x)$ for $x\in\left]0,\frac{\pi}{2}\right[$ then

$$f'(x)=\cos(x) \qquad [\text{Differentiating}]$$

Now $\cos(x)$ is positive for all x in the interval $\left]0,\frac{\pi}{2}\right[$. We have $f'(x)=\cos(x)>0$ for $x\in\left]0,\frac{\pi}{2}\right[$ therefore by theorem (2.18) part (I) the function $\sin(x)$ is increasing in the interval $\left[0,\frac{\pi}{2}\right]$.

Example 40

Show that $\cos(x)$ is decreasing in the interval $[0,\pi]$.

Solution.

Let $f(x)=\cos(x)$ for $x\in\,]0,\pi[$ then

$$f'(x)=-\sin(x) \qquad [\text{Differentiating}]$$

(2.18) Part (I) $f'(x)>0$ for all $x\in\,]a,b[\quad\Rightarrow\quad f$ is **increasing** on $[a,b]$

Because $\sin(x)$ is positive for $x\in\,]0,\pi[$ therefore $-\sin(x)$ is negative in this interval. Hence

$$f'(x)=-\sin(x)<0 \text{ for all } x\in\,]0,\pi[$$

Therefore by theorem (2.18) part (II) the function $f(x)=\cos(x)$ is decreasing in the interval $[0,\pi]$. Remember negative derivative gives a decreasing function.

G4 **Showing Convergence of an Alternate Series**

We can now use the Increasing and Decreasing Theorem (2.18) to apply the alternating series test to a much wider range of series. This is a difficult task because you need to check **all** three conditions of alternating series test (2.17) and in checking these you need to apply theorem (2.18). Additionally you have to know what is meant by the terms increasing and decreasing functions.

Let's go back and do some more examples in this field by using theorem (2.18) to establish condition 3), the decreasing inequality $a_{n+1}<a_n$, of the alternating series test.

Example 41

Show that the following series converges:

$$\sum\left((-1)^{n+1}\ln\left(\frac{n+1}{n}\right)\right)$$

Solution.

Is the given series an alternating series?

Yes because it has the term $(-1)^{n+1}$. We need to use the alternating series test (2.17) which means we have to check all three conditions. Let

$$a_n=\ln\left(\frac{n+1}{n}\right)=\ln\left(1+\frac{1}{n}\right) \qquad \left[\text{Because } \frac{n+1}{n}=\frac{n}{n}+\frac{1}{n}=1+\frac{1}{n}\right]$$

1) For all $n\in\mathbb{N}$ the terms $a_n=\ln\left(1+\frac{1}{n}\right)$ are positive so condition 1) is satisfied.

2) We need to check $\lim_{n\to\infty}(a_n)=0$:

$$\lim_{n\to\infty}(a_n)=\lim_{n\to\infty}\left[\ln\left(1+\frac{1}{n}\right)\right]=\ln(1)=0$$

Hence condition 2) is satisfied. *What else do we need to check?*

3) We need to check that we have a decreasing sequence (a_n) that is $a_{n+1}<a_n$ for all $n\in\mathbb{N}$. *How?*

What is a_{n+1} equal to in this example?

Substituting $n+1$ for n in $a_n = \ln\left(1+\frac{1}{n}\right)$ gives

$$a_{n+1} = \ln\left(1+\frac{1}{n+1}\right)$$

Since

(2.18) Part (II) $f'(x) < 0$ for all $x \in \left]a,b\right[\quad \Rightarrow \quad f$ is **decreasing** on $[a,b]$

$$n+1 > n$$

$$\frac{1}{n+1} < \frac{1}{n} \qquad \left[\text{Because } a > b \text{ implies } \frac{1}{a} < \frac{1}{b}\right]$$

$$\ln\left(1+\frac{1}{n+1}\right) < \ln\left(1+\frac{1}{n}\right) \qquad \left[\begin{array}{l}\text{Because ln is an increasing function,}\\ x < y \text{ implies } \ln(x) < \ln(y)\end{array}\right]$$

We have already shown that the logarithmic function, $\ln(x)$, is an increasing function for $x > 0$ in Example 38. Hence

$$a_{n+1} = \ln\left(1+\frac{1}{n+1}\right) < \ln\left(1+\frac{1}{n}\right) = a_n$$

condition 3) is satisfied.
All three conditions are met therefore by the alternating series test (2.17) the given series $\sum\left((-1)^{n+1}\ln\left(\frac{n+1}{n}\right)\right)$ converges.

Example 42

Show that the series $\sum_{n=2}^{\infty}\left((-1)^n ne^{-n}\right)$ converges.

Solution.
We have an alternating series because

$$\sum_{n=2}^{\infty}\left((-1)^n ne^{-n}\right) = (-1)^2 2e^{-2} + (-1)^3 3e^{-3} + (-1)^4 4e^{-4} + (-1)^5 5e^{-5} + \ldots$$

$$= \frac{2}{e^2} - \frac{3}{e^3} + \frac{4}{e^4} - \frac{5}{e^5} + \ldots \qquad \left[\text{Rewriting } e^{-n} = \frac{1}{e^n}\right]$$

To test this series for convergence we need to apply the alternating series test (2.17).
What is a_n equal to in this case?

It is the general term in the series without $(-1)^n$ hence $a_n = ne^{-n} = \frac{n}{e^n}$.

1) For all $n \geq 2$ (because the series starts at $n = 2$) $a_n > 0$ [Positive]
Hence condition 1) is satisfied. *What else do we need to check?*
2) Check that $\lim_{n\to\infty}(a_n) = 0$

Substituting $a_n = ne^{-n} = \frac{n}{e^n}$ gives

$$\lim_{n\to\infty}(a_n) = \lim_{n\to\infty}\left(\frac{n}{e^n}\right)$$
$$= \lim_{n\to\infty}\left(\frac{1}{e^n}\right) \qquad \left[\text{By L' Hopital's Rule (1.??)}\right]$$
$$= \frac{1}{\lim_{n\to\infty}(e^n)} = 0$$

(1.??) L'Hopitals Rule $\lim_{n\to\infty}\left(\frac{f(n)}{g(n)}\right) = \lim_{n\to\infty}\left(\frac{f'(n)}{g'(n)}\right)$

Therefore condition 2) is satisfied.
3) Next we need to check the inequality $a_{n+1} < a_n$ for $n \geq 2$. *How?*
Since $a_n = ne^{-n}$ we consider the function

$$f(x) = xe^{-x}$$

If we can show that f is a decreasing function then we have the decreasing sequence (a_n) which means $a_{n+1} < a_n$. *But how do we show this?*
By applying the increasing and decreasing theorem (2.18) to the function $f(x) = xe^{-x}$. Differentiating $f(x) = xe^{-x}$ by using the product rule we have

$$f'(x) = \left(xe^{-x}\right)' = (1)e^{-x} + x\left(-e^{-x}\right) \qquad \left[\text{The product rule } (uv)' = u'v + uv'\right]$$
$$= e^{-x} - xe^{-x}$$
$$= e^{-x}(1-x) \qquad \left[\text{Factorizing}\right]$$
$$f'(x) = \frac{1-x}{e^x} \qquad \left[\text{Rewriting } e^{-x} = \frac{1}{e^x}\right]$$

For $x > 1$ the numerator $1 - x < 0$ but the denominator $e^x > 0$ therefore

$$f'(x) = \frac{1-x}{e^x} < 0 \quad \text{for } x > 1$$

Since we have a negative derivative for $x > 1$ so by theorem (2.18) part (II) the function $f(x) = xe^{-x}$ is decreasing for $x > 1$. Therefore

$$a_n = ne^{-n}$$

is decreasing for $n > 1$ and because n is a integer so (a_n) is a decreasing sequence for $n \geq 2$.
All three conditions of the alternating series test are satisfied, so by (2.17) the given series $\sum_{n=2}^{\infty}\left((-1)^n ne^{-n}\right)$ converges.

In the last example the series, $\sum_{n=2}^{\infty}\left((-1)^n ne^{-n}\right)$, started from $n = 2$ but of course the series would also be convergent if we started at $n = 1$. *Why?*

Because at $n=1$ we have a finite value, $-1\times1\times e^{-1}=-e^{-1}$, and since the infinite series $\sum_{n=2}^{\infty}\left((-1)^n ne^{-n}\right)$ converges therefore adding a finite value to the sum does **not** make any difference to the convergence. We can somewhat relax the conditions of the alternating series test given in (2.17) above:

Alternating Series Test (2.17). Let $\sum\left((-1)^{n+1}a_n\right)$ be a series and a_n satisfy the following:

1) $a_n>0$ for all $n\in\mathbb{N}$
2) $\lim_{n\to\infty}(a_n)=0$
3) $a_{n+1}<a_n$ for all $n\in\mathbb{N}$

(2.18) Part (II) $f'(x)<0$ for all $x\in\left]a,b\right[\Rightarrow f$ is **decreasing** on $[a,b]$

then the alternating series $\sum\left((-1)^{n+1}a_n\right)$ converges.

Conditions 1) and 3) can be relaxed to

1) $a_n>0$ eventually

3) $a_{n+1}<a_n$ eventually

That is the terms, a_n, are eventually positive and decreasing. We can rewrite the Alternating Series test as:

Corollary (2.19). Let $\sum\left((-1)^{n+1}a_n\right)$ be a series and a_n satisfy the following:

1) $a_n>0$ for all $n\ge M$ for some natural number M

2) $\lim_{n\to\infty}(a_n)=0$

3) $a_{n+1}<a_n$ for all $n\ge K$ for some natural number K

then the alternating series $\sum\left((-1)^{n+1}a_n\right)$ converges.

Proof. See Exercise 2(g).

It is same as alternating series test (2.17) but conditions 1) and 3) are relaxed to a_n being eventually positive and decreasing. It is possible that a_n in the series is negative or increasing for a finite number of values and eventually it satisfies the given conditions but the series will still converge. This is why we relax conditions 1) and 3).

Example 43

Show that the following series converges:

$$\sum\left((-1)^{n+1}\frac{\ln(n)}{n}\right)$$

Solution.

We have an alternating series so we need to apply the alternating series test or it's corollary above. Let $a_n=\dfrac{\ln(n)}{n}$.

1) For all $n\in\mathbb{N}$ the terms $a_n>0$ [Positive] so condition 1) is satisfied.

2) Need to check condition 2) that is $\lim_{n\to\infty}(a_n)=0$:

$$\lim_{n\to\infty}(a_n) = \lim_{n\to\infty}\left(\frac{\ln(n)}{n}\right)$$

$$= \lim_{n\to\infty}\left(\frac{1/n}{1}\right) \qquad \text{[Applying L' Hopital's Rule]}$$

$$= \lim_{n\to\infty}\left(\frac{1}{n}\right) = 0$$

Hence condition 2) is satisfied.

3) We have to show that the sequence $a_n = \dfrac{\ln(n)}{n}$ is decreasing.

(1.??) L'Hopitals Rule $\lim_{n\to\infty}\left(\dfrac{f(n)}{g(n)}\right) = \lim_{n\to\infty}\left(\dfrac{f'(n)}{g'(n)}\right)$

Let $f(x) = \dfrac{\ln(x)}{x}$ then differentiating this by using the quotient rule we have

$$f'(x) = \left(\frac{\ln(x)}{x}\right)' = \frac{\left(\frac{1}{x}\right)x - \ln(x)(1)}{x^2} \qquad \left[\text{By the Quotient Rule } \left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}\right]$$

$$= \frac{1-\ln(x)}{x^2} \qquad \text{[Simplifying Numerator]}$$

The denominator of $f'(x)$ is x^2 and is therefore positive for $x \neq 0$. The numerator $1-\ln(x) < 0$ [Negative] for

$$\ln(x) > 1 \quad \text{which gives} \quad x > e$$

We know $\ln(e) = 1$ so for $\ln(x) > 1$ we need $x > e$ because ln is an increasing function.

Hence $f'(x) < 0$ for $x > e$. The derivative is negative for $x > e$ so by theorem (2.18) part (II) the function $f(x) = \dfrac{\ln(x)}{x}$ is decreasing for $x > e$. Therefore

$$a_n = \frac{\ln(n)}{n}$$

is a decreasing sequence for $n > e$ and because n is a whole number so (a_n) is a decreasing sequence for $n \geq 3$.

Hence all three conditions of the corollary (2.19) are satisfied therefore the given alternating series $\sum\left((-1)^{n+1}\dfrac{\ln(n)}{n}\right)$ converges.

Note that in Example 43 we used the corollary (2.19) because a_n started decreasing from $n \geq 3$ and **not** from $n = 1$. In fact $a_2 = \dfrac{\ln(2)}{2} > \dfrac{\ln(1)}{1} = a_1$ in Example 43.

G5 **Properties of Alternating Series**

Proposition (2.20).

Let $a_n \in \mathbb{R}^+$ and (a_n) be a decreasing sequence with $\lim_{n\to\infty}(a_n)=0$. Then

$$\left|\sum\left((-1)^{n+1}a_n\right)-S_n\right|\le a_{n+1}$$

where S_n is the nth partial sum given by

$$S_n=\sum_{k=1}^{n}\left((-1)^{k+1}a_k\right)$$

Proof.

Since (a_n) is positive and a decreasing sequence with $\lim_{n\to\infty}(a_n)=0$ therefore it satisfies the three conditions of the alternating series test (2.17) so it converges. Let

$$S=\sum\left((-1)^{n+1}a_n\right)=a_1-a_2+a_3-a_4+a_5-\dots$$

What do we need to prove?

(2.18) Part (II) The derivative of f is negative so it is **decreasing** on $[a,b]$

We are required to prove $|S-S_n|\le a_{n+1}$. *What is nth partial sum, S_n, equal to?*

$$S_n=\sum_{k=1}^{n}\left((-1)^{k+1}a_k\right)=a_1-a_2+a_3-a_4+\dots(-1)^{n+1}a_n$$

Substituting $S=a_1-a_2+a_3-a_4+a_5-\dots$ and $S_n=a_1-a_2+a_3-a_4+\dots(-1)^{n+1}a_n$ into $|S-S_n|$ gives

$$|S-S_n|=\left|\left[a_1-a_2+a_3-\dots(-1)^{n+1}a_n+(-1)^{n+2}a_{n+1}+\dots\right]-\left[a_1-a_2+a_3-a_4+\dots(-1)^{n+1}a_n\right]\right|$$

$$=\left|\left((-1)^{n+2}a_{n+1}\right)+\left((-1)^{n+3}a_{n+2}\right)+\left((-1)^{n+4}a_{n+3}\right)+\left((-1)^{n+5}a_{n+4}\right)+\dots\right| \quad \left[\text{Subtracting Terms}\right]$$

$$=\left|(-1)^{n+2}\right|\left|a_{n+1}-a_{n+2}+a_{n+3}-a_{n+4}+a_{n+5}-\dots\right| \quad \left[\begin{array}{l}\text{Taking Out Common Factor}\\ \text{of } (-1)^{n+2} \text{ and Using } |xy|=|x||y|\end{array}\right]$$

$$=\left|a_{n+1}-a_{n+2}+a_{n+3}-a_{n+4}+a_{n+5}-\dots\right| \quad \left[\text{Because } \left|(-1)^{n+2}\right|=1\right]$$

But how do we show

$$\left|a_{n+1}-a_{n+2}+a_{n+3}-a_{n+4}+a_{n+5}-\dots\right|\le a_{n+1}\,?$$

Rewriting the expression inside the bars:

$$\left|a_{n+1}-a_{n+2}+a_{n+3}-a_{n+4}+a_{n+5}-\dots\right|=\left|a_{n+1}-(a_{n+2}-a_{n+3})-(a_{n+4}-a_{n+5})-\dots\right|$$

ALL the bracketed terms in the Right Hand Side , $(a_{n+2}-a_{n+3})$, $(a_{n+4}-a_{n+5})$ etc are positive because (a_n) is a decreasing sequence. Hence we have the required result.

$$\left|a_{n+1}-(a_{n+2}-a_{n+3})-(a_{n+4}-a_{n+5})-\dots\right|\le a_{n+1}$$

We have proven

$$\left|\sum\left((-1)^{n+1}a_n\right)-S_n\right|\le a_{n+1}$$

■

We prove other properties of the alternating series in Exercise 2(g)

SUMMARY

A series of the form $\sum\left((-1)^{n+1} a_n\right)$ where $a_n \in \mathbb{R}^+$ is an **alternating series**.

Alternating Series Test (2.17). Let $\sum\left((-1)^{n+1} a_n\right)$ be a series and a_n satisfy the following:

1) $a_n > 0$ for all $n \in \mathbb{N}$
2) $\lim_{n\to\infty}(a_n) = 0$
3) $a_{n+1} < a_n$ for all $n \in \mathbb{N}$

then the alternating series $\sum\left((-1)^{n+1} a_n\right)$ converges.

We can relax conditions 1) and 3) to a_n is eventually positive and decreasing.

To establish inequality 3) sometimes involves using the Increasing and Decreasing Theorem (2.18) from Differentiation. The gist of this theorem is that if the derivative is **positive** then we have an **increasing** function but if the derivative is **negative** then we have a **decreasing** function.