# Chapter 2 : Infinite Series

## Section E **Ratio Test**

By the end of this section you will be able to

- understand the proof of the ratio test
- test a series for convergence by applying the ratio test
- appreciate the limitations of the ratio test

## E1 **Proof of the Ratio Test**

The ratio test can be used for testing convergence of a series. It depends on the limiting value of the ratio of (n+1)th term to the nth term.

Ratio Test (2.14). Let $\sum_{n=1}^{\infty}(a_n)$ be a series where $a_n$ is real and positive. Let

$$\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right)=L \quad \text{[Ratio of the (n+1)th term to the nth term]}$$

(I) If $L<1$ then the series $\sum(a_n)$ converges

(II) If $L>1$ then the series $\sum(a_n)$ diverges

(III) If $L=1$ the test fails and we **cannot** conclude whether the series $\sum(a_n)$ converges or diverges.

Note: *What does the Ratio test mean in everyday language?*
The ratio test is divided into three parts. Part (I) says that if the limiting value of (n+1)th term divided by the nth term of a given series is less than 1 then the series converges. *What does part (II) mean?*
It means that if the limiting value of (n+1)th term divided by the nth term of a given series is greater than 1 then the series diverges. *What does part (III) mean?*
If the limiting value of (n+1)th term divided by the nth term of a given series is equal to 1 then we **cannot** say whether the series converges or diverges. The test fails.
The proof of part (I) is difficult because it requires knowledge of inequalities, limit of a sequence, geometric series, comparison test etc. Also from the outset it is difficult to know why the geometric series will play a part in the proof of the ratio test. The only clue is the inequalities $L<1$, $L>1$ and $L=1$ in the statement of the ratio test.
*Proof of (I).* We assume that

$$\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right)=L \quad \text{where } L<1$$

By the definition of a limit of a sequence (1.1) there is a natural number $K$ such that $\forall\varepsilon>0$

$$\left|\frac{a_{n+1}}{a_n}-L\right|<\varepsilon \quad \text{for all } n\geq K$$

(1.1) $\lim_{n\to\infty}(x_n)=L \iff \forall\varepsilon>0\ \exists K\in\mathbb{N}$ such that $\forall n\geq K$

$$|x_n-L|<\varepsilon$$

For $n=K$ we have

$$\left|\frac{a_{K+1}}{a_K} - L\right| < \varepsilon$$

$$\frac{a_{K+1}}{a_K} < L + \varepsilon \qquad \begin{bmatrix}\text{Because } |x - L| < \varepsilon \text{ means} \\ L - \varepsilon < x < L + \varepsilon\end{bmatrix}$$

Let $r = L + \varepsilon$ where $L < 1$ and we choose $\varepsilon > 0$ so that

$$r = L + \varepsilon < 1 \qquad (\dagger)$$

We can always find such an $\varepsilon$ because $L < 1$.
Substituting $r = L + \varepsilon$ into the above inequality we have

$$\frac{a_{K+1}}{a_K} < r$$

$$a_{K+1} < r(a_K) \qquad (*)$$

*What does this inequality mean?*
It means that the (K+1)th term is less than $r$ times the Kth term. Since this inequality (*) holds for all $n \geq K$ so similarly for $n = K+2,\ K+3,\ K+4, \ldots$ we have

$$a_{K+2} < r(a_{K+1}) < r(ra_K) \qquad [\text{By } (*)]$$
$$= r^2 a_K$$
$$a_{K+3} < r(a_{K+2}) < r(r^2 a_K) \qquad [\text{By Above}]$$
$$= r^3 a_K$$
$$a_{K+4} < r(a_{K+3}) < r(r^3 a_K) \qquad [\text{By Above}]$$
$$= r^4 a_K$$

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Adding all these together we have

$$(a_{K+1}) + (a_{K+2}) + (a_{K+3}) + (a_{K+4}) + \ldots < ra_K + r^2 a_K + r^3 a_K + r^4 a_K + \ldots$$
$$= a_K \underbrace{\left(r + r^2 + r^3 + r^4 + \ldots\right)}_{\text{Geometric Series}} \qquad [\text{Factorizing}]$$
$$= a_K \sum_{n=1}^{\infty} (r^n) \qquad (**)$$

Since the modulus of the common ratio $|r| < 1$ (see $(\dagger)$) therefore by (2.3) the geometric series $\sum_{n=1}^{\infty}(r^n)$ converges. Hence the above series $a_K \sum_{n=1}^{\infty}(r^n)$ converges because $a_K$ is a constant.

Since the series with **larger** terms $a_K \sum_{n=1}^{\infty}(r^n)$ converges therefore by the comparison test (2.12) the series (on the Left Hand Side of (**)) with **smaller** terms

$$(a_{K+1}) + (a_{K+2}) + (a_{K+3}) + (a_{K+4}) + \ldots \text{ also converges}$$

(2.3) $\sum(r^n)$ is convergent if $|r| < 1$

(2.12) If $0 \leq a_n \leq b_n$ then $\sum(b_n)$ is convergent $\Rightarrow$ $\sum(a_n)$ is convergent

*What are we trying to prove?*

We need to show that $\sum_{n=1}^{\infty}(a_n)$ converges. *How?*

We have already shown that the series converges from $K+1$, so we can split the infinite series, $\sum_{n=1}^{\infty}(a_n)$, into the first finite $K$ terms plus the remaining terms from $K+1$ onwards. We have

$$\sum_{n=1}^{\infty}(a_n) = \underbrace{a_1 + a_2 + a_3 + a_4 + \ldots + a_K}_{\text{Finite Number of Terms}} + \underbrace{a_{K+1} + a_{K+2} + a_{K+3} + \ldots}_{\text{Converges by Above}}$$

*Does this series converge?*

Yes because we have a finite number of terms plus a series which we have shown converges therefore the whole series, $\sum_{n=1}^{\infty}(a_n)$, converges.

Hence we have shown that if $L<1$ then the series $\sum(a_n)$ converges.

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Try the proofs of parts (II) and (III) for yourself. It does take time to proof something for yourself but it is an excellent way of learning mathematics. If you do get "stuck" then examine the proofs below.

*Proof of (II).* We need to prove that if $L>1$ then the series $\sum(a_n)$ diverges.

Again we assume that

$$\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right) = L \text{ where } L>1$$

By the definition of a limit of a sequence (1.1) there is a natural number $K$ such that $\forall\varepsilon>0$

$$\left|\frac{a_{n+1}}{a_n} - L\right| < \varepsilon \text{ for all } n \geq K$$

$$\frac{a_{n+1}}{a_n} > L-\varepsilon \qquad \begin{bmatrix}\text{Because } |x-L|<\varepsilon \text{ means} \\ L-\varepsilon < x < L+\varepsilon\end{bmatrix}$$

Since $L>1$ we choose $\varepsilon>0$ such that

$$L-\varepsilon>1$$

By substituting this into the above we have

$$\frac{a_{n+1}}{a_n} > 1$$

$$a_{n+1} > a_n > 0 \qquad \text{for all } n \geq K$$

$a_n>0$ because we are considering positive terms. Since positive (n+1)th term is greater than the positive nth term therefore $\lim_{n\to\infty}(a_n)\neq 0$ [Not Zero]. By (2.6) the series $\sum(a_n)$ diverges.

Hence we have shown that if $L>1$ then the series $\sum(a_n)$ diverges. ■

(1.1) $\lim_{n\to\infty}(x_n) = L \iff \forall\varepsilon>0\ \exists K\in\mathbb{N}$ such that $\forall n\geq K \quad |x_n - L| < \varepsilon$

(2.6) If $\lim_{n\to\infty}(a_n)\neq 0$ then $\sum(a_n)$ diverges

*Proof of (III)*. Need to prove that if $L=1$ then the test fails and we **cannot** conclude whether the series $\sum(a_n)$ converges or diverges.

*How do we prove the test fails for* $L=1$ *?*

We can give 2 examples of series one of which converges and the other diverges such that in both cases

$$L=\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right)=1$$

We first give an example of a series which diverges. Consider $a_n=\frac{1}{n}$. *What is* $a_{n+1}$ *equal to?*

Replacing $n$ by $n+1$ gives

$$a_{n+1}=\frac{1}{n+1}$$

$$\frac{a_{n+1}}{a_n}=\left(\frac{1}{n+1}\right)\div\left(\frac{1}{n}\right)$$

$$=\left(\frac{1}{n+1}\right)\times\left(\frac{n}{1}\right) \quad \left[\begin{array}{l}\text{Inverting the Second}\\ \text{Fraction and Multiplying}\end{array}\right]$$

$$=\frac{n}{n+1} \quad [\text{Multiplying}]$$

We have

$$L=\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right)=\lim_{n\to\infty}\left(\frac{n}{n+1}\right) \quad \left[\text{Substituting for } \frac{a_{n+1}}{a_n}\right]$$

$$=\lim_{n\to\infty}\left(\frac{1}{1+1/n}\right) \quad \left[\begin{array}{l}\text{Dividing Numerator and}\\ \text{Denominator by } n\end{array}\right]$$

$$=\frac{1}{1+0}=1$$

With $a_n=\frac{1}{n}$ we have the series $\sum\left(\frac{1}{n}\right)$ which diverges because it is the well established harmonic series.

Now consider $a_n=\frac{1}{n^2}$. *What is* $a_{n+1}$ *equal to?*

$$a_{n+1}=\frac{1}{(n+1)^2}$$

$$\frac{a_{n+1}}{a_n}=\frac{1}{(n+1)^2}\div\left(\frac{1}{n^2}\right)$$

$$=\frac{1}{(n+1)^2}\times\left(\frac{n^2}{1}\right) \quad \left[\begin{array}{l}\text{Inverting the Second}\\ \text{Fraction and Multiplying}\end{array}\right]$$

$$\frac{a_{n+1}}{a_n}=\frac{n^2}{(n+1)^2}=\left(\frac{n}{n+1}\right)^2 \quad \left[\text{Because } \frac{x^2}{y^2}=\left(\frac{x}{y}\right)^2\right]$$

$$L = \lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right) = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^2 \qquad \left[\text{Substituting } \frac{a_{n+1}}{a_n} = \left(\frac{n}{n+1}\right)^2\right]$$

$$= \lim_{n\to\infty}\left(\frac{1}{1+1/n}\right)^2 \qquad \left[\begin{array}{l}\text{Dividing Numerator and}\\ \text{Denominator by } n\end{array}\right]$$

$$= \left(\frac{1}{1+0}\right)^2 = 1$$

The series $\sum(a_n) = \sum\left(\frac{1}{n^2}\right)$ converges because this is the p-series with $p = 2 > 1$.

Hence in each case $L = 1$ and $\sum\left(\frac{1}{n}\right)$ diverges but $\sum\left(\frac{1}{n^2}\right)$ converges. We have shown that the ratio test fails when $L = 1$.

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To determine whether a series converges or not it is generally easier to apply the ratio test then using the comparison test because you don't need to find a series to compare with. The limiting value of the ratio test

$$L = \lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right)$$

is similar to the limiting value of the limit comparison test

$$L = \lim_{n\to\infty}\left(\frac{a_n}{b_n}\right)$$

but with the comparison test you're comparing with another series $\sum(b_n)$ and there are different conditions on $L$.

However it does depend on the series and with attempting plenty of examples in this field you will become familiar with which test to use for a particular series.

Also the limitation of the ratio test is that it fails when $L = 1$.

The ratio test is often called d'Alembert ratio test after the French mathematician Jean d'Alembert. d'Alembert was born in Paris in 1717 and he studied mathematics at 'College des Quatre Nations' which had an excellent mathematics library. d'Alembert studied many subjects such as law, medicine, theology, etc he even qualified as an advocate but it was his love for mathematics that had priority.

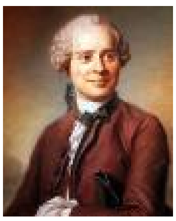

Fig 1 Jean d'Alembert

He worked at the Paris Academy of Science and the French Academy all his life. His work on limits lead to the ratio test for the convergence of a series. The ratio test is named after him 'd'Alembert's ratio test'.

Next we apply the ratio test to particular examples. The application of this method is generally straightforward but can involve a large number of algebraic steps.

## E2 **Applying the Ratio Test**

Example 25

Determine whether the following series

$$\sum\left(\frac{1}{n!}\right)$$

converges or diverges.

Solution.

*Can we apply the ratio test?*

Yes because we only need the terms of the series to be positive and since $1/n! > 0$ therefore we can use the ratio test. *What does $n!$ mean?*

$$n! = 1\times2\times3\times4\times\ldots\times n$$
$$(n+1)! = 1\times2\times3\times4\times\ldots\times n\times(n+1)$$

*Let $a_n = \frac{1}{n!}$ then what is $a_{n+1}$ equal to?*

Replacing $n$ with $n+1$ into $a_n = \frac{1}{n!}$ gives

$$a_{n+1} = \frac{1}{(n+1)!}$$

*What do we need to find in order to use the ratio test?*

The limiting value of

$$\frac{a_{n+1}}{a_n} = \frac{1}{(n+1)!}\div\left(\frac{1}{n!}\right) \qquad \left[\text{Substituting for } a_n \text{ and } a_{n+1}\right]$$

$$= \frac{1}{(n+1)!}\times\left(\frac{n!}{1}\right) \qquad \left[\begin{array}{l}\text{Inverting the Second}\\ \text{Fraction and Multiplying}\end{array}\right]$$

$$= \frac{n!}{(n+1)!}$$

$$= \frac{1\times2\times3\times4\times\ldots\times n}{1\times2\times3\times4\times\ldots\times n\times(n+1)} = \frac{1}{n+1} \qquad \left[\begin{array}{l}\text{Cancelling All the}\\ \text{Common Factors}\end{array}\right]$$

We have

$$L = \lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right)$$

$$= \lim_{n\to\infty}\left(\frac{1}{n+1}\right) = 0 \qquad \left[\text{Substituting } \frac{a_{n+1}}{a_n} = \frac{1}{n+1}\right]$$

Since $L = 0 < 1$ therefore by the ratio test

(2.14) (I) If $L < 1$ then the series $\sum(a_n)$ converges

the given series $\sum\left(\frac{1}{n!}\right)$ converges.

Remember the ratio test says that if $L < 1$ then the series under consideration converges. We **only** need to check the value of $L$. The ‘meat’ is in evaluating $L$.

Example 26

Discuss the convergence or divergence of

$$\sum_{n=0}^{\infty}(n!)$$

Solution.

*How can we test the given series for convergence?*

Since the nth term does **not** converge to 0, that is

$$\lim_{n\to\infty}(n!)\neq 0$$

we can conclude by (2.6) that $\sum_{n=0}^{\infty}(n!)$ diverges. We can confirm this by applying the ratio test as follows:

Let $a_n = n!$ then $a_{n+1} = (n+1)!$ Substituting these into $a_{n+1}/a_n$ gives

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{n!}$$

$$= \frac{1\times 2\times 3\times 4\times \ldots \times n\times (n+1)}{1\times 2\times 3\times 4\times \ldots \times n} = n+1 \qquad \begin{bmatrix}\text{Cancelling}\\ \text{Common Factors}\end{bmatrix}$$

Hence

$$L = \lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right) = \lim_{n\to\infty}(n+1) = +\infty$$

Since $L > 1$ therefore by the ratio test

(2.14) (II) If $L > 1$ then the series $\sum(a_n)$ diverges

the given series $\sum_{n=0}^{\infty}(n!)$ diverges.

The next example is not difficult but just involves a lot of algebra. Of course if you do not know the result of the hint, $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$, then the question of testing the given series for convergence is difficult.

Example 27

Show that the following series

$$\sum\left(\frac{n!}{n^n}\right)$$

converges. [Hint: $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$].

Solution.

*Can we apply the ratio test?*

Yes because $\frac{n!}{n^n}$ is positive for all $n\in\mathbb{N}$. Let $a_n = \frac{n!}{n^n}$. *What is $a_{n+1}$ equal to?*

Replacing $n$ with $n+1$ gives

(2.6) If $\lim_{n\to\infty}(a_n)\neq 0$ then $\sum(a_n)$ diverges

$$a_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$$

Let

$$L = \lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right) \qquad (*)$$

Let's investigate the terms inside the brackets.

$$\frac{a_{n+1}}{a_n} = \left(\frac{(n+1)!}{(n+1)^{n+1}}\right) \div \left(\frac{n!}{n^n}\right) \qquad \left[\text{Substituting for } a_{n+1} \text{ and } a_n\right]$$

$$= \left(\frac{(n+1)!}{(n+1)^{n+1}}\right) \times \left(\frac{n^n}{n!}\right) \qquad \left[\begin{array}{l}\text{Inverting the Second}\\ \text{Fraction and Multiplying}\end{array}\right]$$

$$= \frac{(n+1)!\,n^n}{(n+1)^{n+1}\,n!}$$

$$= \frac{(n+1)\,n^n}{(n+1)^n (n+1)} \qquad \left[\text{Because } \frac{(n+1)!}{n!} = n+1\right]$$

$$= \left(\frac{n}{n+1}\right)^n \qquad \left[\text{Cancelling } (n+1)\text{'s}\right]$$

$$= \left(\frac{n+1}{n}\right)^{-n} \qquad \left[\text{Remember } \left(\frac{x}{y}\right)^n = \left(\frac{y}{x}\right)^{-n}\right]$$

$$= \left(1+\frac{1}{n}\right)^{-n} \qquad \left[\text{Because } \frac{n+1}{n} = \frac{n}{n} + \frac{1}{n} = 1 + \frac{1}{n}\right]$$

$$\frac{a_{n+1}}{a_n} = \frac{1}{\left(1+\frac{1}{n}\right)^n} \qquad \left[\text{Rewriting } x^{-n} = \frac{1}{x^n}\right]$$

Substituting $\frac{a_{n+1}}{a_n} = \frac{1}{\left(1+\frac{1}{n}\right)^n}$ into (*) we have

$$L = \lim_{n\to\infty}\left[\frac{1}{\left(1+\frac{1}{n}\right)^n}\right]$$

$$= \frac{1}{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} < 1 \qquad \text{[By Hint: } \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e\text{]}$$

Since $L < 1$ therefore by the ratio test

(2.14) (I) If $L < 1$ then the series $\sum(a_n)$ converges

the given series $\sum\left(\frac{n!}{n^n}\right)$ converges.

Remember for using the ratio test we need to evaluate $L$. Once we have a particular value of $L$ then we can decide on the question of convergence of a given series.

Example 28

Determine whether the following series converges

$$\sum_{n=1}^{\infty}\left(\frac{n+1}{2n-1}\right)$$

Solution

Let $a_n=\dfrac{n+1}{2n-1}$. *What is* $a_{n+1}$ *equal to?*

Replacing $n$ with $n+1$ in $a_n=\dfrac{n+1}{2n-1}$ gives

$$a_{n+1}=\frac{(n+1)+1}{2(n+1)-1}$$

$$=\frac{n+2}{2n+1} \qquad \left[\text{Simplifying}\right]$$

Substituting these into $a_{n+1}/a_n$ gives

$$\frac{a_{n+1}}{a_n}=\left(\frac{n+2}{2n+1}\right)\div\left(\frac{n+1}{2n-1}\right)$$

$$=\left(\frac{n+2}{2n+1}\right)\times\left(\frac{2n-1}{n+1}\right)$$

$$=\frac{(n+2)(2n-1)}{(2n+1)(n+1)}=\frac{2n^2+3n-2}{2n^2+3n+1} \qquad \left[\text{Expanding Brackets}\right]$$

We have

$$L=\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right)$$

$$=\lim_{n\to\infty}\left(\frac{2n^2+3n-2}{2n^2+3n+1}\right) \qquad \left[\text{Substituting for } \frac{a_{n+1}}{a_n}\right]$$

$$=\lim_{n\to\infty}\left(\frac{2+3/n-2/n^2}{2+3/n+1/n^2}\right) \qquad \left[\begin{array}{l}\text{Dividing Numerator}\\ \text{and Denominator by } n^2\end{array}\right]$$

$$L=1$$

*Does the given series* $\sum_{n=1}^{\infty}\left(\frac{n+1}{2n-1}\right)$ *converge or diverge?*

By applying the ratio test with $L=1$

(2.14) (III) If $L=\lim_{n\to\infty}\left(\dfrac{a_{n+1}}{a_n}\right)=1$ the test fails

we **cannot** answer the question whether the series converges or diverges. *How can we test the given series?*

Well we can find the limit of the nth term of the given series

$$\lim_{n\to\infty}\left(\frac{n+1}{2n-1}\right)=\lim_{n\to\infty}\left(\frac{1+1/n}{2-1/n}\right)\quad\left[\begin{array}{l}\text{Dividing Numerator}\\ \text{and Denominator by } n\end{array}\right]$$

$$=\frac{1}{2}$$

Since the nth term converges to $1/2 \neq 0$ [Not Zero] therefore by (2.6) the given series $\sum_{n=1}^{\infty}\left(\frac{n+1}{2n-1}\right)$ diverges.

For Example 28 above we should have noticed from the outset that the nth term does **not** converge to zero therefore the series diverges. Remember the ratio test does **not** work when $L=1$. We have to apply some other test in this case.

Example 29

Let $(x_n)$ be a real sequence. Show that if $\forall n \in \mathbb{N}$

$$0 < x_{n+1} < x_n$$

then $\sum (x_n)^n$ converges.

Solution

*Can we apply the ratio test?*

Yes because $x_n$ is positive for all the natural numbers $n$. Let $a_n = (x_n)^n$ then $a_{n+1} = (x_{n+1})^{n+1}$. We have

$$\frac{a_{n+1}}{a_n}=\frac{(x_{n+1})^{n+1}}{(x_n)^n}$$

$$=\frac{(x_{n+1})^n x_{n+1}}{(x_n)^n}\quad\left[\text{Using the Rules of Indices}\right]$$

$$=\left(\frac{x_{n+1}}{x_n}\right)^n x_{n+1}\qquad(\dagger)$$

Since $0 < x_{n+1} < x_n$ therefore

$$\frac{x_{n+1}}{x_n} < 1$$

Let $r=\frac{x_{n+1}}{x_n} < 1$ and substituting this into $(\dagger)$ we have $\frac{a_{n+1}}{a_n} = r^n x_{n+1}$

$$L=\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right)$$

$$=\lim_{n\to\infty}\left(r^n x_{n+1}\right)\quad\left[\text{Substituting } \frac{a_{n+1}}{a_n}=r^n x_{n+1}\right]$$

*How do we evaluate* $\lim_{n\to\infty}\left(r^n x_{n+1}\right)$*?*

Since $r<1$ therefore $\lim_{n\to\infty}\left(r^n\right)=0$.

(2.6) If $\lim_{n\to\infty}(a_n) \neq 0$ then $\sum (a_n)$ diverges

Also because the sequence $(x_n)$ is a monotonic (decreasing) sequence and is bounded by $x_1$ therefore by the Monotonic Sequence Theorem (1.??) $(x_n)$ converges. Let

$$M = \lim_{n\to\infty}(x_n)$$

then from above we have

$$L = \lim_{n\to\infty}(r^n x_{n+1}) = \underbrace{\lim_{n\to\infty}(r^n)}_{=0 \text{ because } r<1} \lim_{n\to\infty}(x_{n+1}) = 0 \times M = 0$$

Since $L = 0 < 1$ by the ratio test

(2.14) (I) If $L < 1$ then the series $\sum(a_n)$ converges

the given series $\sum(x_n)^n$ converges provided that $0 < x_{n+1} < x_n$ for all $n \in \mathbb{N}$.

There are lots of examples in Exercise 2e for you to tackle. Try to attempt **all** questions in this exercise. The only way you will become familiar with the ratio test is to do a large number of questions even if your lecturer does not assign any. This is the **only** way of learning mathematics.

SUMMARY

Ratio Test (2.14). Let $\sum_{n=1}^{\infty}(a_n)$ be a series where $a_n$ is real and positive. Let

$$\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right) = L \quad \text{[Ratio of the (n+1)th term to the nth term]}$$

(I) If $L < 1$ then the series $\sum(a_n)$ converges

(II) If $L > 1$ then the series $\sum(a_n)$ diverges

(III) If $L = 1$ the test fails and we **cannot** conclude whether the series $\sum(a_n)$ converges or diverges.

(1.??) A bounded monotonic sequence converges