SECTION E **Properties of Composite Functions**

By the end of this section you will be able to
- understand what is meant by the identity function
- prove properties of inverse function
- prove properties of composition of functions

E1 Properties of the Identity Function

What do you think the term identity means?
The word identity in everyday language means being the same. *What does 'identity function' mean?*
The function which has the **same** input and output. The definition of the identity function is:

Definition (3.12). Let A be any set. The function $f: A \to A$ given by

$$f(x) = x \qquad \text{for all } x \text{ in } A$$

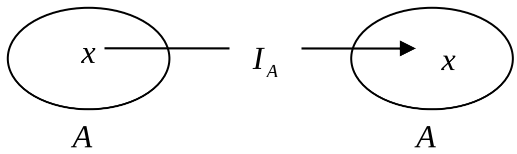
is called the **identity function** on the set A.

The identity function on a set A is normally denoted by I_A or just I if the set A is clear. Hence we have for all x in A the identity function $I_A: A \to A$ satisfying

$$I_A(x) = x$$

The identity function I_A transforms x back to x. This can be illustrated by

Fig 31



If x^2 *is in* A *then what is* $I_A(x^2)$ *equal to?*

$I_A(x^2) = x^2$. If $x+y$ is in A then $I_A(x+y) = x+y$. *What is* $I_A(\Sigma)$ *equal to?*

$$I_A(\Sigma) = \Sigma \quad \text{provided } \Sigma \text{ is in set } A$$

The identity function transforms back to itself. The input is same as the output.
The identity function has many important properties that we use. In this section we state and prove some of these properties.

Proposition (3.13).
Let f be any function, $f: A \to B$, then

$$f \circ I_A = f = I_B \circ f$$

Proof. Let x be an arbitrary element in A which means it is in the domain of f because we are given $f: A \to B$. *Is it in the domain of* $f \circ I_A$*?*

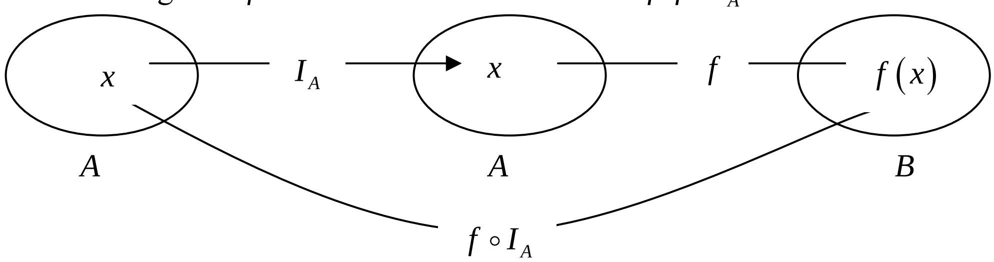


Fig 32

Yes because $I_A : A \to A$ and so $f \circ I_A : A \to B$. We have

$$\begin{aligned}(f \circ I_A)(x) &= f(I_A(x)) && [\text{Definition of } \circ] \\ &= f(x) && [\text{Because } I_A(x) = x]\end{aligned}$$

Since $(f \circ I_A)(x) = f(x)$ therefore $f \circ I_A = f$.

Next we show $I_B \circ f = f$. *Is the element $x \in A$ in the domain of $I_B \circ f$?*

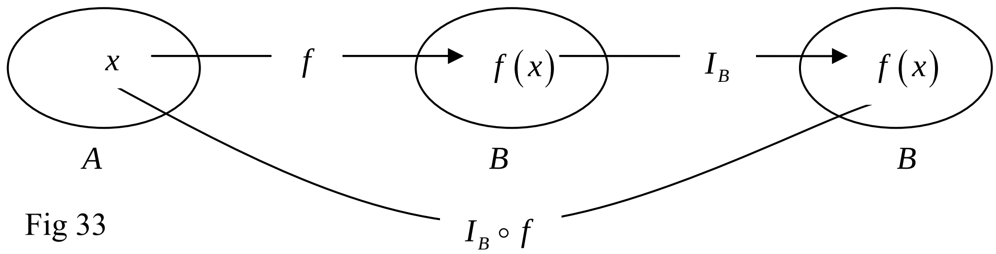


Fig 33

Yes because $f : A \to B$ and therefore $I_B \circ f : A \to B$. We have

$$\begin{aligned}(I_B \circ f)(x) &= I_B(f(x)) && [\text{Definition of } \circ] \\ &= f(x) && [\text{Because } I_B(f(x)) = f(x)]\end{aligned}$$

Since $(I_B \circ f)(x) = f(x)$ therefore $I_B \circ f = f$.

Remember both I_A and I_B are identity functions so their arguments x and $f(x)$ do **not** change by applying these functions.

Hence we have our required result, $f \circ I_A = f = I_B \circ f$.

■

Prior to this section we have used lower case letters such as f, g and h to represent functions but for the identity function we use upper case I to represent this function.

E2 Properties of the Inverse Function

Proposition (3.14).

Let $f : A \to B$ be a bijective function. Then

$$f^{-1} \circ f = I_A \text{ and } f \circ f^{-1} = I_B$$

where I_A is the identity function on A and I_B is the identity function on B.

Proof.

We prove $f^{-1} \circ f = I_A$ and leave the proof of $f \circ f^{-1} = I_B$ as a question in Exercise 3(e).

Since f is bijective therefore the inverse function f^{-1} exists. *Why?*

Because from Section C we have

(3.5) $\quad f$ has an inverse $\Leftrightarrow$ f is bijective

Let x be an arbitrary element in the set A and $f(x) = y$. Therefore y is in B because $f : A \to B$.

$$\begin{aligned}(f^{-1} \circ f)(x) &= f^{-1}(f(x)) && [\text{Definition of } \circ] \\ &= f^{-1}(y)\end{aligned}$$

Is y in the domain of f^{-1}?

Yes because $f^{-1}: B \to A$ that is the domain of f^{-1} is B and $y = f(x)$. By the definition of the inverse function we have $f^{-1}(y) = x$ and therefore

$$\left(f^{-1} \circ f\right)(x) = f^{-1}(y) \qquad \text{[From Above]}$$
$$= x = I_A(x) \qquad \text{[Because the identity function } I_A(x) = x\text{]}$$

Since $\left(f^{-1} \circ f\right)(x) = I_A(x)$ therefore we have the required result, $f^{-1} \circ f = I_A$. ■

Proposition (3.15).

Let the functions $f : A \to B$ and $g : B \to A$ satisfy **both**

$$g \circ f = I_A \text{ and } f \circ g = I_B$$

Then the function g is unique and $g = f^{-1}$.

Comment. For $g = f^{-1}$ we need the function f to be bijective. *Why?*

Because from Section C we have

(3.5) f has an inverse $\Leftrightarrow$ f is bijective

First we prove that f is bijective and then $g = f^{-1}$. Hence first we show that f is injective and surjective.

Proof.

How do we prove that f is injective (one to one)?

By (3.2) we are required to show that for all $x,\ y$ in A we have

$$f(x) = f(y) \text{ implies } x = y$$

Let both $x,\ y$ be in A and $f(x) = f(y)$. Then

$$x = I_A(x) \qquad \text{[Identity Function]}$$
$$= (g \circ f)(x) \qquad \text{[Because we are given } g \circ f = I_A\text{]}$$
$$= g(f(x)) \qquad \text{[By definition of } \circ\text{]}$$
$$= g(f(y)) \qquad \text{[Because } f(x) = f(y)\text{]}$$
$$= (g \circ f)(y) \qquad \text{[By definition of } \circ\text{]}$$
$$= I_A(y) = y \qquad \text{[Because we are given } g \circ f = I_A\text{]}$$

Hence $x = y$ therefore by (3.2) we conclude that f is injective.

Next we prove f is surjective. *How?*

Let y be in B (codomain of f) and then show that there is an element, call it x, in A (domain of f) such that $f(x) = y$. We have

$$y = I_B(y) \qquad \text{[Identity Function]}$$
$$= (f \circ g)(y) \qquad \text{[Because we are given } f \circ g = I_B\text{]}$$
$$= f(g(y)) \qquad \text{[Definition of } \circ\text{]}$$

(3.2) f is an **injection** $\Leftrightarrow$ $f(x) = f(y)$ implies $x = y$

Since we are given $g : B \to A$ therefore $g(y)$ is in A. Let $x = g(y)$ and therefore we have found an element, $x = g(y)$, in A, such that $f(x) = y$. Hence the function f is surjective.
Because f is both injective and surjective therefore it is bijective. By
(3.5) f has an inverse $\Leftrightarrow$ f is bijective.
we can say the function f has an inverse f^{-1}. *Have we completed the proof?*
No because we still need to show that g is unique and $g = f^{-1}$. *How do we prove this?*
By contradiction. Suppose they are **not** equal, that is $g \neq f^{-1}$. This means there is another function besides f^{-1} with the above stated properties:

$$g \circ f = I_A \text{ and } f \circ g = I_B$$

The domain of both $f^{-1} : B \to A$ and $g : B \to A$ is the set B. There must be an element say y in B such that

$$g(y) \neq f^{-1}(y) \qquad \left[\text{Because functions } g \neq f^{-1}\right]$$

Since the function f is injective (one to one) therefore

$$f(g(y)) \neq f\left(f^{-1}(y)\right) \qquad \text{[Not Equal]}$$

But

$$\begin{aligned} f(g(y)) &= (f \circ g)(y) && [\text{Definition of } \circ] \\ &= I_B(y) = y && [\text{Because we are given } f \circ g = I_B] \end{aligned}$$

and

$$\begin{aligned} f\left(f^{-1}(y)\right) &= \left(f \circ f^{-1}\right)(y) && [\text{Definition of } \circ] \\ &= I_B(y) = y && \left[\text{Because by (3.14) we have } f \circ f^{-1} = I_B\right] \end{aligned}$$

We have $f(g(y)) = f\left(f^{-1}(y)\right) = y$. Contradiction because earlier we stated they were **not equal**, that is $f(g(y)) \neq f\left(f^{-1}(y)\right)$.
Hence we have proved our result $g = f^{-1}$.

■

Can you see any relationship between the above propositions (3.14) and (3.15)?
In (3.15) we have proven that the inverse function f^{-1} which satisfies proposition (3.14) is unique:

$$f^{-1} \circ f = I_A \text{ and } f \circ f^{-1} = I_B$$

f^{-1} is the **only function** with this property for a given function f. *What does this mean?*
Let $f : A \to B$ and x be in the domain A then

$$\left(f^{-1} \circ f\right)(x) = x$$

Similarly if y is in B then $\left(f \circ f^{-1}\right)(y) = y$.
Note that $g = f^{-1}$ only if **both** the following conditions are satisfied:

$$g \circ f = I_A \text{ and } f \circ g = I_B$$

Remember bijective functions are crucial in the discussion of inverse functions but they are also preserved under the composition of functions as we will show in the exercises.

E3 Properties of the Composite Functions

Next we prove an extremely important property of function composition. We prove that the composition of functions is associative.

Proposition (3.16)

Let $h:A\to B,\quad g:B\to C$ and $f:C\to D$ be functions. Then

$$(f\circ g)\circ h=f\circ(g\circ h)$$

We can illustrate the functions by:

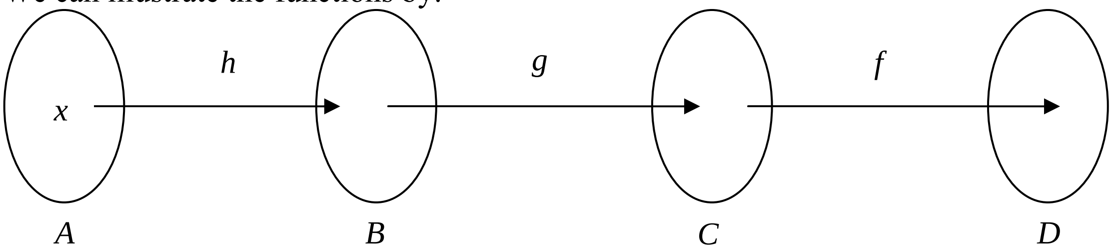


Fig 34

Proof.

The domain of both $(f\circ g)\circ h$ and $f\circ(g\circ h)$ is the set A. Let x be an arbitrary element in A. Then

$$\begin{aligned}((f\circ g)\circ h)(x)&=(f\circ g)(h(x)) && [\text{By Definition of } \circ]\\ &=f(g(h(x))) && [\text{By Definition of } \circ]\end{aligned}$$

Also

$$\begin{aligned}(f\circ(g\circ h))(x)&=f(g\circ h(x)) && [\text{Definition of } \circ]\\ &=f(g(h(x))) && [\text{Definition of } \circ]\end{aligned}$$

Since for the arbitrary element x we have

$$((f\circ g)\circ h)(x)=(f\circ(g\circ h))(x)$$

therefore we have our required result:

$$(f\circ g)\circ h=f\circ(g\circ h)$$

■

SUMMARY

The identity function on a set A is denoted by I_A and has the property

$$I_A(x)=x$$

for all x in A. The input and output of the function stay the same.

Let $f:A\to B$ and $g:B\to A$ be functions which satisfy **both**

$$g\circ f=I_A \text{ and } f\circ g=I_B$$

Then $g=f^{-1}$ and is unique.

Composition of functions is associative:

$$(f\circ g)\circ h=f\circ(g\circ h)$$