Exercise 5b

1. Prove that the infimum of a set is unique.
2. Prove proposition (5.6).
3. Let $S = \{x \mid a \le x \le b \text{ and } x \in \mathbb{R}\}$. Prove that $\inf(S) = a$.
4. A real number L of a non-empty subset, S, of $\mathbb{R}$ is the infimum of $S \Leftrightarrow$
   (i) For all $s \in S$, $s \ge L$
   (ii) If $m > L$ then there exists $y \in S$ such that $m > y$.
5. Let S and T be non-empty subsets of $\mathbb{R}$ that are bounded below by real numbers. Let the set $S + T$ be defined as
   $$S + T = \{s + t \mid s \in S \text{ and } t \in T\}$$
   Prove that $\inf(S + T) = \inf(S) + \inf(T)$.
6. Let S be a non-empty subset of $\mathbb{R}$ which is bounded above and let $k \in \mathbb{R}$. We define the set
   $$k + S = \{k + s \mid s \in S\}$$
   Show that $\sup(k + S) = k + \sup(S)$.
7. Let S be a non-empty subset of $\mathbb{R}$ which is bounded above. Show that $u \in \mathbb{R}$ is an upper bound of S if and only if $t \in \mathbb{R}$ and $t > u \Rightarrow t \notin S$.
8. Let S be a non-empty bounded set of real numbers and $k \in \mathbb{R}$. We define the set
   $$kS = \{ks \mid s \in S\}$$
   Prove the following results:
   (a) If $k > 0$ then
   (i) $\sup(kS) = k\sup(S)$
   (ii) $\inf(kS) = k\inf(S)$
   (b) If $k < 0$ then
   (i) $\sup(kS) = k\inf(S)$
   (ii) $\inf(kS) = k\sup(S)$
9. Let A and B be non-empty subsets of $\mathbb{R}$ bounded above and below respectively. Prove that if for all $a \in A$ and $b \in B$
   $$a \le b$$
   then $\sup(A) \le \inf(B)$.
10. Let S and T be sets of positive numbers bounded above. Let the set ST be defined by
    $$ST = \{st \mid s \in S \text{ and } t \in T\}$$
    Prove that $\sup(ST) = \sup(S)\sup(T)$.
11. Let S and T be non-empty subsets of $\mathbb{R}$ bounded above. Let the set ST be defined by
    $$ST = \{st \mid s \in S \text{ and } t \in T\}$$
    Show that in general $\sup(ST) \ne \sup(S)\sup(T)$.

Solutions 5b

1. Similar to the proof of (5.4) (i). Assume there are two L_1 and L_2 and show that they are equal, $L_1 = L_2$.
2. Similar to the proof of (5.5).
3. Similar to the proof of proposition (5.8). Suppose $\inf(S) > a$ and $\inf(S) < a$ and then arrive at a contradiction.
4. Apply proposition (5.6) with $m = L + \varepsilon$ and an appropriate $\varepsilon > 0$.
5. Similar to the proof of proposition (5.9).
6. Use proof by contradiction with the supposition $\sup(k+S) < k + \sup(S)$ and then use proposition (5.7) to produce the contradiction.
7. This is an if and only if proof so you need to go both ways, $\Rightarrow$ and $\Leftarrow$.
8. Use proof by contradiction in each case. Apply proposition (5.7) or the proposition in Question ?? to produce the contradiction.
9. *Proof*. Suppose $\sup(A) > \inf(B)$. By the proposition in Question ??, $\exists b \in B$ such that $\sup(A) > b$. Since $b < \sup(A)$, by proposition (5.7) $\exists a \in A$ such that $b < a$ .This contradicts $a \le b$. Hence $\sup(A) \le \inf(B)$.
10. Again use proof by contradiction and proposition (5.7).
11. Consider the sets:

$$S = \{ s \mid s < -1 \text{ and } s \in \mathbb{R} \}$$
$$T = \{ t \mid t < -2 \text{ and } t \in \mathbb{R} \}$$

Then $ST = \{ st \mid st > 2 \}$. Hence the set ST is **not** bounded above and so **cannot** have a supremum.