Exercise **3(b)**

Remember ***one to one*** *functions are alternatively called* ***injective*** *functions. Also* ***onto*** *functions are called* ***surjective*** *functions.*

1. Let $f:\mathbb{R}\to\mathbb{R}$ be given by

(a) $f(x)=x$ (b) $f(x)=3x+1$ (c) $f(x)=x^3$

For each of these functions state whether f is
(i) injective (one to one)
(ii) surjective (onto)
In each case show that the functions are indeed injective and/or surjective.
[The function in part (a) given by $f(x)=x$ is called the **identity function**].

2. Decide and show whether the functions given by the formulae in question 1 are
(i) injective (one to one)
(ii) surjective (onto)
if the domain and codomain are both the set of all integers, $\mathbb{Z}$.

3. Decide whether the function $f:\mathbb{R}\to\mathbb{R}$ given by $f(x)=\pi$ is one to one**,** onto, both or neither. How can you change the function so that it is onto? Write this new function.

4. (i) Let the function $f:\mathbb{R}\to\mathbb{R}$ be given by

$$f(x)=x^2+1$$

Show that this function is **neither** injective **nor** surjective.
(ii) What is the largest codomain so that f is surjective (onto). Write this new function.

5. (i) Let the function $f:\mathbb{R}^+\to\mathbb{R}$ be given by

$$f(x)=x^2-1$$

Show that this function is injective but **not** surjective.
(ii) What is the largest codomain so that f is surjective (onto). Write this new function.

6. Let the function $g:\mathbb{N}\to\mathbb{N}$ be defined by

$$g(n)=2^n$$

Prove that g is one to one but **not** onto.

7. Let the function $f:\mathbb{R}\to\mathbb{R}^+$ be defined by

$$f(x)=\begin{cases} x & \text{if } x\geq 0 \\ -x & \text{if } x<0 \end{cases}$$

Prove that f is **not** injective but it is surjective.
If the codomain is $\mathbb{R}$ then show that f is neither injective nor surjective.

8. Consider the following functions:
(a) $f_1 : [-2, \ 2] \to \mathbb{R}$ (b) $f_2 : [0, \ 5] \to \mathbb{R}$ (c) $f_3 : [-1, \ 1] \to [0, \ 1]$
where each is given by the same formula

$$f(x) = x^2$$

Decide and prove whether each of these functions is injective and/or surjective.

9. Consider Dirichlet's function $f : \mathbb{R} \to \{0, \ 1\}$

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ 1 & \text{if } x \text{ is rational} \end{cases}$$

Prove that f is **not** injective but is surjective.

10. What is a domain and largest codomain of the sin function for it to be both injective and surjective?

11. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^n$ where $n \in \mathbb{N}$. For what values of n is f
(a) one to one? (b) onto?

12. The successor function $s : \mathbb{N} \to \mathbb{N}$ is defined by

$$s(n) = n + 1$$

Prove that s is injective but **not** surjective.

13. Let $f : \mathbb{R}^+ \to \mathbb{R}$ be defined by $f(x) = 2x^2 - 2x + 1$. Prove that f is one to one but **not** onto.

14. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = x + \frac{1}{x} \quad (x \neq 0)$$

Decide whether f is injective and/or surjective and prove your conclusions.

Solutions to Exercise 3b

1. (a) Both injective (one to one) and surjective (onto).
(b) Both injective (one to one) and surjective (onto).
(c) Both injective (one to one) and surjective (onto).

2. (a) Both injective (one to one) and surjective (onto).
(b) Injective (one to one) but **not** surjective (onto).
(c) Injective (one to one) but **not** surjective (onto).

3. Neither. New function $g : \mathbb{R} \to \{\pi\}$ given by $g(x) = \pi$.

4. (ii) The largest codomain is $B = \{x \mid x \in \mathbb{R} \text{ and } x \geq 1\}$

5. (ii) The largest codomain is $B = \{x \mid x \in \mathbb{R} \text{ and } x \geq -1\}$

8. (a) f_1 is neither injective nor surjective.
(b) f_2 is injective but not surjective.
(c) f_3 is not injective but is surjective.

10. Domain is $[0,\ 2\pi]$ and codomain is $[-1,\ 1]$.

11. When n is odd.