Exercise 2f

Throughout this exercise $\sum$ represents $\sum_{n=1}^{\infty}$.

1. Determine whether the following improper integrals converge or diverge. If they converge determine its value.

(a) $\int_1^{\infty}\left(\frac{dx}{x^2}\right)$ (b) $\int_1^{\infty}\left(\frac{dx}{x^3}\right)$ (c) $\int_1^{\infty}\left(\frac{dx}{x^n}\right)$

2. Determine whether the following improper integrals converge or diverge. If they converge determine its value.

(a) $\int_0^{\infty}\left(\frac{x}{x^2+1}\right)dx$ (b) $\int_1^{\infty}\left(\frac{1}{x^2+1}\right)dx$ (c) $\int_0^{\infty}\left(xe^{-x^2}\right)dx$

3. Discuss the convergence or divergence of the following series:

(a) $\sum_{n=0}^{\infty}\left(e^{-n}\right)$ (b) $\sum\left(\frac{1}{2n-1}\right)$ (c) $\sum\left(\frac{n}{n^2+1}\right)$

(d) $\sum\left(\frac{1}{n^2+4}\right)$ (e) $\sum\left(\frac{1}{n^2+3n+2}\right)$

4. Test the following series for convergence:

(a) $\sum\left(n^2e^{-n^3}\right)$ (b) $\sum\left(\frac{e^{-\sqrt{n}}}{2\sqrt{n}}\right)$ (c) $\sum\left(ne^{-n}\right)$

(d) $\sum\left(2ne^{-n^2}\right)$

5. (a) Discuss the convergence or divergence of $\sum\left(\frac{\ln(n)}{n}\right)$.

(b) Show that $\sum_{n=2}^{\infty}\left(\frac{1}{n(\ln(n))^p}\right)$ converges for $p>1$.

6. (i) Test the series $\sum\left(\frac{1}{n^2+1}\right)$ by the integral, comparison and ratio tests.

(ii) Compare and contrast between the tests for the series in part (i).

7. Show that

$$\frac{9}{8}<\sum\left(\frac{1}{n^3}\right)<\frac{5}{4}$$

Solutions

1. (a) Converges with value 1.
   (b) Converges with value ½.
   (c) Converges with value $\frac{1}{n-1}$.
2. (a) Diverges
   (b) Converges with value $\frac{\pi}{4}$.
   (c) Converges with value $\frac{1}{2}$.
3. (a) Converges with the analogous improper integral equal to 1.
   (b) Diverges.
   (c) Diverges.
   (d) Converges with the analogous improper integral equal to $\pi/4$.
   (e) Converges with the analogous improper integral equal to $\ln(1)-\ln\left(\frac{2}{3}\right)$.
4. (a) Converges with the analogous improper integral equal to $1/3e$.
   (b) Converges with the analogous improper integral equal to $1/e$.
   (c) Converges with the analogous improper integral equal to $1/e$.
   (d) Converges with the analogous improper integral equal to $1/e$.
5. (a) Diverges.
   (b) Evaluating the improper integral gives $\int_2^\infty \left(\frac{dx}{x\left[\ln(x)\right]^p}\right) = \frac{1}{(p-1)\left[\ln(2)\right]^{p-1}}$ therefore the series converges for $p>1$.
6. (i) Converges.
   (ii) Ratio test fails. Comparison test is easier to apply than the integral test.
7. Since $\sum_{n=1}^{\infty}\left(\frac{1}{n^3}\right) = 1+\frac{1}{8}+\sum_{n=3}^{\infty}\left(\frac{1}{n^3}\right)$ therefore $\sum_{n=1}^{\infty}\left(\frac{1}{n^3}\right) > \frac{9}{8}$. Next show that $\sum_{n=3}^{\infty}\left(\frac{1}{n^3}\right) < \frac{1}{8}$.