Complete Solutions to Exercise 5(f)

1. By using the "Monotone Convergence Theorem" show that $\lim_{n\to\infty}(x_n)=0$ where $x_n=\frac{1}{n}$.

Proof. To use the "Monotone Convergence Theorem" we have to show that the sequence $x_n=\frac{1}{n}$ is bounded and monotone. We can try looking at the first few terms of the sequence by substituting $n=1,\ 2,\ 3,\ 4$ into $x_n=\frac{1}{n}$:

$$x_1=\frac{1}{1}=1$$
$$x_2=\frac{1}{2}$$
$$x_3=\frac{1}{3}$$
$$x_4=\frac{1}{4}$$

We first show that the sequence $x_n=\frac{1}{n}$ is decreasing (monotone). This means we need to show $x_{n+1}\le x_n$ for all $n\in\mathbb{N}$. Need to compare $x_{n+1}=\frac{1}{n+1}$ and $x_n=\frac{1}{n}$.
Since

$$n+1>n \quad\Leftrightarrow\quad \frac{1}{n+1}<\frac{1}{n}$$

Hence $x_{n+1}=\frac{1}{n+1}<\frac{1}{n}=x_n$. Therefore we have $x_{n+1}<x_n$ for all $n\in\mathbb{N}$ which means that the given sequence is decreasing. We also need to show that (x_n) is bounded. *How?*
Show that $x_n\le 1$ by induction. Clearly the result is true for $x_1=1\le 1$. √
Assume the result is true for $n=k$ that is $x_k\le 1$. Required to prove the result for $n=k+1$.

$$x_{k+1}=\frac{1}{k+1}<\frac{1}{k}\le 1$$

Hence $x_{k+1}\le 1$ therefore by induction we have proven that for all $n\in\mathbb{N}$ we have $x_n\le 1$ which means the given sequence is bounded.

Therefore $x_n=\frac{1}{n}$ is a bounded monotone (decreasing) sequence so by the Monotone Convergence Theorem the sequence converges. *What does it converge to?*
Converges to

$$\lim_{n\to\infty}\left(\frac{1}{n}\right)=\inf\left\{\frac{1}{n}\,\middle|\, n\in\mathbb{N}\right\}$$
$$=\inf\left\{1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{5},\cdots\right\}=0$$

■

2. By using the "Monotone Convergence Theorem" show that $\lim_{n\to\infty}(x_n)=0$ where $x_n=\dfrac{1}{\sqrt{n}}$.

Proof. To use the "Monotone Convergence Theorem" we have to show that the given sequence $x_n=\dfrac{1}{\sqrt{n}}$ is bounded and monotone. We can try looking at the first few terms of the sequence by substituting $n=1,\ 2,\ 3,\ 4$ into $x_n=\dfrac{1}{\sqrt{n}}$:

$$x_1=\frac{1}{\sqrt{1}}=1$$
$$x_2=\frac{1}{\sqrt{2}}$$
$$x_3=\frac{1}{\sqrt{3}}$$
$$x_4=\frac{1}{\sqrt{4}}=\frac{1}{2}$$

We first show that the sequence $x_n=\dfrac{1}{\sqrt{n}}$ is decreasing (monotone) by proving $x_{n+1}=\dfrac{1}{\sqrt{n+1}}<\dfrac{1}{\sqrt{n}}=x_n$. Since

$$n+1>n \quad\Leftrightarrow\quad \sqrt{n+1}>\sqrt{n} \quad\Leftrightarrow\quad \frac{1}{\sqrt{n+1}}<\frac{1}{\sqrt{n}}$$

Hence $x_{n+1}=\dfrac{1}{\sqrt{n+1}}<\dfrac{1}{\sqrt{n}}=x_n$. Therefore we have $x_{n+1}<x_n$ for all $n\in\mathbb{N}$ which means that the given sequence is decreasing. We also need to show that (x_n) is bounded. *How?*

Show that $x_n\le 1$ by induction. Clearly the result is true for $x_1=1\le 1$. √

Assume the result is true for $n=k$ that is $x_k=\dfrac{1}{\sqrt{k}}\le 1$. Required to prove the result for $n=k+1$. Since $k+1>k$ we have

$$x_{k+1}=\frac{1}{\sqrt{k+1}}<\frac{1}{\sqrt{k}}\le 1$$

Hence $x_{k+1}\le 1$ therefore by induction we have proven that for all $n\in\mathbb{N}$ we have $x_n\le 1$ which means the given sequence (x_n) is bounded.

Therefore $x_n=\dfrac{1}{\sqrt{n}}$ is bounded monotone (decreasing) sequence so by the

Monotone Convergence Theorem the sequence converges. *What does it converge to?*
By (5.25) we have

$$\lim_{n\to\infty}\left(\frac{1}{\sqrt{n}}\right)=\inf\left\{\frac{1}{\sqrt{n}}\,\middle|\, n\in\mathbb{N}\right\}$$
$$=\inf\left\{1,\frac{1}{\sqrt{2}},\frac{1}{\sqrt{3}},\frac{1}{2},\frac{1}{\sqrt{5}},\cdots\right\}=0$$

■

3. Let $x_1=1$ and $x_{n+1}=\frac{1}{5}\left(x_n+3\right)$. Prove that $\lim_{n\to\infty}\left(x_n\right)=\frac{3}{4}$.

Proof. We examine the first few terms of the sequence by substituting $n=1,\ 2,\ 3,\ 4,\ 5$ into $x_{n+1}=\frac{1}{5}\left(x_n+3\right)$:

$$x_1=1 \qquad [\text{Given}]$$
$$x_2=\frac{1}{5}\left(x_1+3\right)=\frac{1}{5}\left(1+3\right)=\frac{4}{5}=0.8 \qquad \begin{bmatrix}\text{Substituting}\\ n=1 \text{ and } x_1=1\end{bmatrix}$$
$$x_3=\frac{1}{5}\left(x_2+3\right)=\frac{1}{5}\left(\frac{4}{5}+3\right)=\frac{19}{25}=0.76 \qquad \begin{bmatrix}\text{Substituting}\\ n=2 \text{ and } x_2=4/5\end{bmatrix}$$
$$x_4=\frac{1}{5}\left(x_3+3\right)=\frac{1}{5}\left(\frac{19}{25}+3\right)=\frac{94}{125}=0.752 \qquad \begin{bmatrix}\text{Substituting}\\ n=3 \text{ and } x_3=19/25\end{bmatrix}$$
$$x_5=\frac{1}{5}\left(x_4+3\right)=\frac{1}{5}\left(0.752+3\right)=0.7504 \qquad \begin{bmatrix}\text{Substituting}\\ n=4 \text{ and } x_4=0.752\end{bmatrix}$$

We need to show the given sequence $\left(x_n\right)$ is a bounded monotone sequence. We first prove that $\left(x_n\right)$ is decreasing (monotone) that is $x_{n+1}\le x_n$ for all $n\in\mathbb{N}$.
How?
By using mathematical induction. Clearly $x_2=0.8\le 1=x_1$ that is $x_2\le x_1$. √
Assume

$$x_k\le x_{k-1} \quad\text{or}\quad x_k-x_{k-1}\le 0 \qquad (*)$$

Required to prove $x_{k+1}\le x_k$ or equivalently

$$x_{k+1}-x_k\le 0$$

We have

$$x_{k+1}-x_k=\frac{1}{5}\left(x_k+3\right)-\frac{1}{5}\left(x_{k-1}+3\right)$$
$$=\frac{1}{5}\left(x_k-x_{k-1}\right) \qquad [\text{Simplifying}]$$
$$\le 0 \qquad [\text{Because by } (*) \text{ we have } x_k\le x_{k-1}]$$

Therefore $x_{k+1}-x_k\le 0$ which gives $x_{k+1}\le x_k$. Hence by induction we have $x_{n+1}\le x_n$ for all $n\in\mathbb{N}$ which means the given sequence $\left(x_n\right)$ is decreasing.
Need to show $\left(x_n\right)$ is bounded that is for all $n\in\mathbb{N}$ we have $x_n\ge 3/4$.

(5.25) If (x_n) is bounded and decreasing sequence then $\lim_{n\to\infty}(x_n)=\inf\{x_n \mid n\in\mathbb{N}\}$

How can we prove this result?

Again by induction. We have $x_1 = 1 \ge \frac{3}{4}$ √

Assume $x_k \ge \frac{3}{4}$. Required to prove $x_{k+1} \ge \frac{3}{4}$. Consider

$$\begin{aligned} x_{k+1} &= \frac{1}{5}(x_k+3) \\ &\ge \frac{1}{5}\left(\frac{3}{4}+3\right)=\frac{3}{4} \end{aligned}$$

Hence by induction we have our result $x_n \ge \frac{3}{4}$ for all $n\in\mathbb{N}$ which means that the given sequence (x_n) is bounded.

We conclude that the sequence (x_n) is a bounded monotone sequence therefore by the Monotone Convergence Theorem the sequence (x_n) converges and moreover

$$\begin{aligned} \lim_{n\to\infty}(x_n) &= \inf\{x_n \mid n\in\mathbb{N}\} \\ &= \inf\{1,\ 0.8,\ 0.76,\ 0.752,\ 0.7504,\cdots\} \\ &= 0.75 = \frac{3}{4} \end{aligned}$$

Therefore we have proved our required result, that is $\lim_{n\to\infty}(x_n)=\frac{3}{4}$.

■

4. Prove that the sequence $x_n=\sqrt{1-\frac{1}{n}}$ is convergent by using the Monotone Convergence Theorem.

Proof. We can evaluate the first few terms of the sequence $x_n=\sqrt{1-\frac{1}{n}}$ by substituting $n=1,\ 2,\ 3,\ 4,\cdots$ and working to 3 decimal places:

$x_1=\sqrt{1-\frac{1}{1}}=0$ $\left[\text{Substituting } n=1 \text{ into } x_n=\sqrt{1-\frac{1}{n}}\right]$

$x_2=\sqrt{1-\frac{1}{2}}=\frac{1}{\sqrt{2}}=0.707$ $\left[\text{Substituting } n=2 \text{ into } x_n=\sqrt{1-\frac{1}{n}}\right]$

$x_3=\sqrt{1-\frac{1}{3}}=\sqrt{\frac{2}{3}}=0.816$ $\left[\text{Substituting } n=3 \text{ into } x_n=\sqrt{1-\frac{1}{n}}\right]$

$x_4=\sqrt{1-\frac{1}{4}}=\sqrt{\frac{3}{4}}=0.866$ $\left[\text{Substituting } n=4 \text{ into } x_n=\sqrt{1-\frac{1}{n}}\right]$

Need to show that the sequence $x_n = \sqrt{1-1/n}$ is increasing. *How?*

By showing $x_{n+1} = \sqrt{1-\frac{1}{n+1}} \geq \sqrt{1-\frac{1}{n}} = x_n$ for all $n \in \mathbb{N}$. Since

$$n+1 > n \quad \Leftrightarrow \quad \frac{1}{n+1} < \frac{1}{n} \qquad \left[\text{Remember if } a > b \text{ then } \frac{1}{a} < \frac{1}{b}\right]$$

$$\Leftrightarrow \quad -\frac{1}{n+1} > -\frac{1}{n} \qquad \left[\begin{array}{l}\text{Multiplying by } -1 \text{ and Applying} \\ a < b \text{ and } c < 0 \text{ then } ca > cb\end{array}\right]$$

$$\Leftrightarrow \quad 1-\frac{1}{n+1} > 1-\frac{1}{n} \qquad [\text{Adding } 1]$$

$$\Leftrightarrow \quad \sqrt{1-\frac{1}{n+1}} > \sqrt{1-\frac{1}{n}} \qquad [\text{Taking Square Root}]$$

Hence $x_{n+1} = \sqrt{1-\frac{1}{n+1}} > \sqrt{1-\frac{1}{n}} = x_n$ for all $n \in \mathbb{N}$ which means that (x_n) is an increasing (monotone) sequence.

To use the Monotone Convergence Theorem we also need to prove that the given sequence (x_n) is bounded. *How?*

Required to prove that $x_n \leq 1$. Since $n \geq 1$ therefore

$$1-\frac{1}{n} \leq 1 \quad \Leftrightarrow \quad \sqrt{1-\frac{1}{n}} \leq \sqrt{1} = 1$$

and so we have $x_n = \sqrt{1-\frac{1}{n}} \leq 1$ for all $n \in \mathbb{N}$ which means that (x_n) is bounded. Because the given sequence (x_n) is bounded and monotone therefore by the Monotone Convergence Theorem we conclude that (x_n) is convergent.

5. Let (x_n) be a real sequence defined by

$$x_1 = 2 \quad \text{and} \quad x_{n+1} = \frac{x_n}{2} + \frac{3}{2x_n} \qquad (☺)$$

Prove that the sequence (x_n) defined in (☺) is monotone and bounded. Also prove that $\lim_{n\to\infty}(x_n) = \sqrt{3}$.

Proof. Very similar to Example 27.

We can evaluate the first few terms of the sequence (x_n) by substituting $n = 1,\ 2,\ 3,\ 4,\ \cdots$ into $x_{n+1} = \frac{x_n}{2} + \frac{3}{2x_n}$ and working to 2 decimal places:

$x_1 = 2$ [Given]

$x_2 = \frac{x_1}{2} + \frac{3}{2x_1} = \frac{2}{2} + \frac{3}{4} = 1.75$ [Substituting $n = 1$ and $x_1 = 2$]

$x_3 = \frac{x_2}{2} + \frac{3}{2x_2} = \frac{1.75}{2} + \frac{3}{(2 \times 1.75)} = 1.73$ [Substituting $n = 2$ and $x_2 = 1.75$]

$x_4 = \frac{x_3}{2} + \frac{3}{2x_3} = \frac{1.73}{2} + \frac{3}{(2 \times 1.73)} = 1.73$ [Substituting $n = 3$ and $x_3 = 1.73$]

How do we show the sequence (x_n) is monotonic and bounded?

We first prove that the given sequence (x_n) is bounded. *How?*

We show that for all $n \in \mathbb{N}$

$$x_n \ge \sqrt{3}$$

By (☺) we have

$$x_{n+1} = \frac{x_n}{2} + \frac{3}{2x_n}$$

$$= \frac{(x_n)^2 + 3}{2x_n} \quad \text{[Common Denominator]}$$

$$= \frac{(x_n)^2 + (\sqrt{3})^2}{2x_n} \quad \left[\text{Rewriting } 3 = (\sqrt{3})^2\right]$$

$$\ge \frac{2x_n\sqrt{3}}{2x_n} \quad \begin{bmatrix} \text{Using } a^2 + b^2 \ge 2ab \text{ on Numerator} \\ (x_n)^2 + (\sqrt{3})^2 \ge 2(x_n)\sqrt{3} \end{bmatrix}$$

$$= \sqrt{3} \quad \left[\text{Cancelling Out Common Factors } 2x_n\right]$$

Hence $x_{n+1} \ge \sqrt{3}$. Since $x_1 = 2 \ge \sqrt{3}$, hence for all $n \in \mathbb{N}$

$$x_n \ge \sqrt{3} \qquad (*)$$

The given sequence (x_n) is bounded.

We also need to show that the given sequence (x_n) is decreasing which means that for all $n \in \mathbb{N}$ we have

$$x_{n+1} \le x_n$$

How do we show this result?

We use (*) and show that $x_{n+1} - x_n \le 0$.

$$x_{n+1} - x_n = \underbrace{\frac{x_n}{2} + \frac{3}{2x_n}}_{=x_{n+1}} - x_n$$

$$= \frac{(x_n)^2 + 3 - 2(x_n)^2}{2x_n} \qquad \text{[Common Denominator]}$$

$$= \frac{3 - (x_n)^2}{2x_n} \qquad \text{[Simplifying]}$$

$$\le 0 \qquad \begin{bmatrix} \text{Because by (*) we have} \\ x_n \ge \sqrt{3} \text{ which implies } (x_n)^2 \ge 3 \end{bmatrix}$$

Hence $x_{n+1} - x_n \le 0$ which gives our required result $x_{n+1} \le x_n$ for all $n \in \mathbb{N}$. Therefore the given sequence (x_n) is decreasing (monotonic). Hence by the monotonic convergence theorem the sequence (x_n) converges.

Let $x = \lim_{n\to\infty}(x_n)$ then by proposition (5.26) we have $x = \lim_{n\to\infty}(x_{n+1})$.

Using these to evaluate the limit of the sequence given in (☺):

(5.26) $\lim_{n\to\infty}(x_n) = \lim_{n\to\infty}(x_{n+1})$

$$x = \lim_{n\to\infty}(x_{n+1}) = \frac{\lim_{n\to\infty}(x_n)}{2} + \frac{3}{2\lim_{n\to\infty}(x_n)} = \frac{x}{2} + \frac{3}{2x}$$

Solving this equation we have

$$x = \frac{x}{2} + \frac{3}{2x}$$

$$2x^2 = x^2 + 3 \qquad \text{[Multiplying by } 2x\text{]}$$

$$x^2 = 3 \qquad \text{[Simplifying]}$$

$$x = \pm\sqrt{3} \qquad \text{[Solving]}$$

$$x = \sqrt{3}, \quad x = -\sqrt{3}$$

$x = \lim_{n\to\infty}(x_n) = \sqrt{3}$ or $x = \lim_{n\to\infty}(x_n) = -\sqrt{3}$. *But which one of these is the limiting value of the given sequence?*

Since $x = \lim_{n\to\infty}(x_n) \ge 0$ because $x_n \ge 0$ for all $n \in \mathbb{N}$ therefore

$x = \lim_{n\to\infty}(x_n) = \sqrt{3}$.

■

6. Let (x_n) be a real sequence defined by

$$x_1 = 1 \quad \text{and} \quad x_{n+1} = \sqrt{1 + x_n}$$

Show that the sequence (x_n) is monotone and bounded. Also determine $\lim_{n\to\infty}(x_n)$.

Solution

We can evaluate the first few terms of the sequence (x_n) by substituting $n=1,\ 2,\ 3,\ 4,\ \cdots$ into $x_{n+1}=\sqrt{1+x_n}$ and working to 3 decimal places:

$$x_1 = 1 \qquad [\text{Given}]$$

$$x_2=\sqrt{1+x_1}=\sqrt{1+1}=\sqrt{2}=1.414 \qquad [\text{Substituting } n=1 \text{ and } x_1=1]$$

$$x_3=\sqrt{1+x_2}=\sqrt{1+\sqrt{2}}=1.554 \qquad \left[\text{Substituting } n=2 \text{ and } x_2=\sqrt{2}\right]$$

$$x_4=\sqrt{1+x_3}=\sqrt{1+1.554}=1.598 \qquad [\text{Substituting } n=3 \text{ and } x_3=1.554]$$

We need to show that the given sequence (x_n) is increasing (monotone) by establishing the inequality $x_{n+1}\ge x_n$ for all $n\in\mathbb{N}$. *How?*

By using induction. Clearly $x_2=1.414\ge 1=x_1$ which means that $x_2\ge x_1$.

Assume

$$x_k \ge x_{k-1} \qquad (\dagger)$$

Required to prove that $x_{k+1}\ge x_k$.

From (†) we have

$$x_k \ge x_{k-1}$$

$$1+x_k \ge 1+x_{k-1} \qquad [\text{Adding } 1]$$

$$\sqrt{1+x_k} \ge \sqrt{1+x_{k-1}} \qquad \left[\text{If } a\ge b\ge 0 \text{ then } \sqrt{a}\ge\sqrt{b}\right]$$

Using this result on the last line we have

$$x_{k+1}=\sqrt{1+x_k}\ge\sqrt{1+x_{k-1}}=x_k$$

We have shown $x_{k+1}\ge x_k$. Hence by induction we have $x_{n+1}\ge x_n$ for all $n\in\mathbb{N}$.

We also need to show that (x_n) is bounded. By examining the first four terms of the sequence

$x_1=1,\ x_2=1.414,\ x_3=1.554$ and $x_4=1.598$ it seems $x_n\le 1.7$. We need to prove $x_n\le 1.7$ for all $n\in\mathbb{N}$. *How do we prove this result?*

Again by using induction. The result holds for $n=1$ that is $x_1=1\le 1.7$. √

Assume the result is true for $n=k$:

$$x_k \le 1.7 \qquad (*)$$

Need to prove the result for $n=k+1$:

$$\begin{aligned} x_{k+1} &= \sqrt{1+x_k} \\ &\le \sqrt{1+1.7} \qquad [\text{By } (*)] \\ &= \sqrt{2.7}=1.643\le 1.7 \end{aligned}$$

Hence $x_{k+1}\le 1.7$. Therefore by induction we conclude $x_n\le 1.7$ for all $n\in\mathbb{N}$ which means that the given sequence (x_n) is bounded. Since (x_n) is a bounded and monotone (increasing) sequence therefore by the Monotone Convergence Theorem the sequence (x_n) converges. *But what value does it converge to?*

Let $x=\lim_{n\to\infty}(x_n)$ then by proposition (5.26) we have $x=\lim_{n\to\infty}(x_{n+1})$. Using these to evaluate the limit of the sequence:

$$x = \lim_{n\to\infty}(x_{n+1}) = \sqrt{1+\lim_{n\to\infty}(x_n)} = \sqrt{1+x}$$

Solving this equation we have

$$x = \sqrt{1+x}$$

$$x^2 = 1+x \qquad \text{[Squaring Both Sides]}$$

$$x^2 - x - 1 = 0 \qquad [\text{Subtracting } 1+x]$$

Solving for x gives

$$x = \frac{-1 \pm \sqrt{(-1)^2+4}}{2} = \frac{-1\pm\sqrt{5}}{2}$$

$$= \frac{-1+\sqrt{5}}{2}, \quad \frac{-1-\sqrt{5}}{2}$$

$$= 1.618, \quad -0.618$$

We have two values of the limit, $x = \lim_{n\to\infty}(x_n) = 1.618$ or $x = \lim_{n\to\infty}(x_n) = -0.618$. *But what is* $\lim_{n\to\infty}(x_n)$ *equal to?*

Since for all $n \in \mathbb{N}$ we have $x_n \geq 0$ therefore

$$x = \lim_{n\to\infty}(x_n) = 1.618$$

(5.26) $\lim_{n\to\infty}(x_n) = \lim_{n\to\infty}(x_{n+1})$

7. Let (x_n) be a real sequence defined by

$$x_1 = 1 \text{ and } x_{n+1} = \sqrt{2+x_n}$$

Show that the sequence (x_n) is monotone and bounded. Also determine $\lim_{n\to\infty}(x_n)$.

Solution.

We can evaluate the first few terms of the sequence (x_n) by substituting $n = 1, 2, 3, 4, \cdots$ into $x_{n+1} = \sqrt{2+x_n}$ and working to 3 decimal places:

$x_1 = 1$ [Given]

$x_2 = \sqrt{2+x_1} = \sqrt{2+1} = \sqrt{3} = 1.732$ [Substituting $n=1$ and $x_1 = 1$]

$x_3 = \sqrt{2+x_2} = \sqrt{2+\sqrt{3}} = 1.932$ [Substituting $n=2$ and $x_2 = \sqrt{3}$]

$x_4 = \sqrt{2+x_3} = \sqrt{2+1.932} = 1.983$ [Substituting $n=3$ and $x_3 = 1.932$]

We need to show that the given sequence (x_n) is increasing (monotone) by establishing the inequality $x_{n+1} \geq x_n$ for all $n \in \mathbb{N}$. *How?*

By using induction. Clearly $x_2 = 1.732 \geq 1 = x_1$ which means that $x_2 \geq x_1$.

Assume

$$x_k \geq x_{k-1} \qquad (\dagger)$$

Required to prove that $x_{k+1} \geq x_k$.

From (†) we have

$$x_k \ge x_{k-1}$$

$$2 + x_k \ge 2 + x_{k-1} \qquad [\text{Adding } 2]$$

$$\sqrt{2+x_k} \ge \sqrt{2+x_{k-1}} \qquad [\text{If } a \ge b \ge 0 \text{ then } \sqrt{a} \ge \sqrt{b}]$$

Using this result on the last line we have

$$x_{k+1} = \sqrt{2+x_k} \ge \sqrt{2+x_{k-1}} = x_k$$

Hence by induction we have $x_{n+1} \ge x_n$ for all $n \in \mathbb{N}$. We also need to show that (x_n) is bounded. By examining the first four terms of the sequence $x_1 = 1,\ x_2 = 1.732,\ x_3 = 1.932$ and $x_4 = 1.983$ it seems $x_n \le 2$. We need to prove $x_n \le 2$ for all $n \in \mathbb{N}$. *How do we prove this result?*

Again by using induction. The result holds for $n = 1$ that is $x_1 = 1 \le 2$.

Assume the result is true for $n = k$:

$$x_k \le 2 \qquad (*)$$

Need to prove the result for $n = k+1$:

$$\begin{aligned} x_{k+1} &= \sqrt{2+x_k} \\ &\le \sqrt{2+2} \qquad [\text{By } (*)] \\ &= \sqrt{4} = 2 \end{aligned}$$

Hence $x_{k+1} \le 2$. Therefore by induction we have $x_n \le 2$ for all $n \in \mathbb{N}$ and so

(5.26) $\lim\limits_{n\to\infty}(x_n) = \lim\limits_{n\to\infty}(x_{n+1})$

we conclude that the given sequence (x_n) is bounded. Since (x_n) is bounded and monotone (increasing) therefore by the Monotone Convergence Theorem the sequence (x_n) converges. *But what value does it converge to?*

Let $x = \lim\limits_{n\to\infty}(x_n)$ then by proposition (5.26) we have $x = \lim\limits_{n\to\infty}(x_{n+1})$. Using these to evaluate the limit of the sequence:

$$x = \lim_{n\to\infty}(x_{n+1}) = \sqrt{2 + \lim_{n\to\infty}(x_n)} = \sqrt{2+x}$$

Solving this equation we have

$$\begin{aligned} x &= \sqrt{2+x} \\ x^2 &= 2 + x \qquad [\text{Squaring Both Sides}] \\ x^2 - x - 2 &= 0 \qquad [\text{Subtracting } 1 + x] \\ (x-2)(x+1) &= 0 \\ x = 2 &\text{ or } x = -1 \end{aligned}$$

We have two values of the limit, $x = \lim\limits_{n\to\infty}(x_n) = 2$ or $x = \lim\limits_{n\to\infty}(x_n) = -1$. *But what is* $\lim\limits_{n\to\infty}(x_n)$ *equal to?*

Since for all $n \in \mathbb{N}$ we have $x_n \ge 0$ therefore

$$x = \lim_{n\to\infty}(x_n) = 2$$

8. Prove the Monotone Convergence Theorem (5.23) for a **bounded decreasing** sequence. (A **bounded monotone** sequence of real numbers is **convergent**).

Proof. Let (x_n) be a bounded decreasing sequence. Therefore the set consisting of the sequence (x_n) is bounded below which means it has a infimum (Greatest Lower Bound) call it L. By proposition (5.6) part (ii) there is an element say $y = x_{N_0+1}$ of the given sequence such that

$$L \le x_{N_0+1} < L + \varepsilon$$

for an arbitrary $\varepsilon > 0$. Since (x_n) is a **decreasing** sequence we have for all $n > N_0$

$$x_n \le x_{N_0+1} < L + \varepsilon \qquad (*)$$

By proposition (5.6) part (i) we also have for all $n \in \mathbb{N}$

$$L - \varepsilon < L \le x_n \qquad (**)$$

By combining these inequalities, (*) and (**), we have for all $n > N_0$ the inequality

$$L - \varepsilon < x_n < L + \varepsilon$$

This means that $|x_n - L| < \varepsilon$ for all $n > N_0$ which implies the sequence (x_n) converges to the limit L. Hence a bounded monotone sequence converges.

■

(5.6) A real number L is the infimum of a non empty set $S \Leftrightarrow$
(i) For all $x \in S$ we have $L \le x$
(ii) For every $\varepsilon > 0$ there is an element, y, in S such that $L \le y < L + \varepsilon$.

9. Prove Proposition (5.25) which says that if (x_n) is a bounded decreasing sequence of real numbers then

$$\lim_{n\to\infty}(x_n) = \inf\{x_n \mid n \in \mathbb{N}\}$$

where inf is the infimum (Greatest Lower Bound) of the set.
Proof. The proof is given in solution to Question 8 above because the L in the above is given by

$$L = \lim_{n\to\infty}(x_n) = \inf\{x_n \mid n \in \mathbb{N}\}$$

■