Complete Solutions to Exercise 1(d)

1.

(a) *Proof*. We assume n and m are even. By definition (1.1) they can be written as

$$n = 2a \text{ and } m = 2b$$

where a and b are integers. Consider their addition $n+m$:

$$\begin{aligned} n+m &= 2a+2b \\ &= 2(a+b) \qquad [\text{Factorizing}] \end{aligned}$$

We have $n+m$ is of the form 2(An Integer). By applying definition (1.1) in the $\Leftarrow$ direction we conclude that $n+m$ is even.

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(b) *Proof*. We assume n and m are even. By definition (1.1) they can be written as

$$n = 2a \text{ and } m = 2b$$

where a and b are integers. Consider their subtraction $n-m$:

$$\begin{aligned} n-m &= 2a-2b \\ &= 2(a-b) \qquad [\text{Factorizing}] \end{aligned}$$

We have $n-m$ is of the form 2(An Integer). By applying definition (1.1) in the $\Leftarrow$ direction we conclude that $n-m$ is even.

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(c) *Proof*. We assume n and m are odd. By definition (1.3) they can be written as

$$n = 2a+1 \text{ and } m = 2b+1$$

where a and b are integers. Consider $n-m$:

$$\begin{aligned} n-m &= (2a+1)-(2b+1) \\ &= 2a-2b \\ &= 2(a-b) \qquad [\text{Factorizing}] \end{aligned}$$

We have $n-m$ is of the form 2(An Integer). By applying definition (1.1) in the $\Leftarrow$ direction we conclude that $n-m$ is even whenever n and m are odd.

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(d) *Proof*. Let n be an odd number then by (1.3) there is an integer m such that $n = 2m+1$. Consider n^2:

$$\begin{aligned} n^2 &= (2m+1)^2 \\ &= (2m+1)(2m+1) \\ &= 4m^2+4m+1 = 2(2m^2+2m)+1 \qquad [\text{Rewriting } 4 = 2(2)] \end{aligned}$$

We have $n^2 = 2(\text{An Integer})+1$. By applying definition (1.3) in the $\Leftarrow$ direction we conclude that n^2 is odd.

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(e) *Proof*. Let n be even. Then by definition (1.1) this can be written as

$$n = 2a \text{ where } a \text{ is an integer}$$

Let m be odd then by (1.3) this can be written as

$$m = 2b+1 \text{ where } b \text{ is an integer}$$

Consider $n+m$:

$$n+m = \underbrace{2a}_{=a} + \underbrace{2b+1}_{=m} = 2(a+b)+1 \quad [2(\text{An Integer})+1]$$

(1.1) n is an even number $\Leftrightarrow$ $n = 2m$ where m is an integer

(1.3) n is an odd number $\Leftrightarrow$ $n = 2m+1$ where m is an integer

We have $n+m$ is 2(Integer) +1 therefore by (1.3) $n+m$ is odd.

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(f) *Proof*. Let n be an odd number then by (1.3) there is an integer a such that $n=2a+1$. Similarly let m be an odd number then there is an integer b such that $m=2b+1$. Consider their product nm:

$$\begin{aligned} nm &= (2a+1)(2b+1) \\ &= 4ab+2a+2b+1 \\ &= 2(2ab+a+b)+1 \quad [2(\text{An Integer})+1] \end{aligned}$$

We have $nm = 2(\text{Integer})+1$. By applying definition (1.3) in the $\Leftarrow$ direction we conclude that the product nm is odd.

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(g) *Proof*. Since m is even we can write this as

$$m=2k \text{ where } k \text{ is an integer}$$

The product nm is given by

$$nm = n(2k) = 2kn$$

Hence nm is a multiple of 2 therefore by definition (1.1) we conclude that nm is even.

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2. (i) n is odd $\Rightarrow$ $n+1$ is even

Proof. We assume n is odd. Since n and 1 are odd therefore by proposition (1.4) we have $n+1$ is even.

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(ii) For any integer n we have $n(n+1)$ is even

Proof. If n is even then by the above proposition in Question 1(g) we have $n(n+1)$ is even. However if n is odd then by the above proposition 2(i) we have $n+1$ is even. Hence again by the above proposition in Question 1(g) we have $n(n+1)$ is even.

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3. If n is odd then n^3-1 is even

Proof. By the proposition in Question 1(d) we have n is odd $\Rightarrow$ n^2 is odd. Similarly by the proposition in Question 1(f) we have n^2 is odd $\Rightarrow$ nn^2 is odd. Hence $nn^2 = n^3$ is odd. Since n^3 and 1 are odd therefore by the proposition in Question 1(c) we have n^3-1 is even. This is what was required.

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4. (a) We need to prove $a \mid 0$.

Proof. Since $a\times 0=0$ therefore by definition (1.5) we have $a \mid 0$.

(b) We need to prove $a \mid a$.

Proof. Since $a\times 1=a$ therefore by definition (1.5) we have $a \mid a$.

(1.1) n is an even number $\Leftrightarrow$ $n=2m$ where m is an integer
(1.3) n is an odd number $\Leftrightarrow$ $n=2m+1$ where m is an integer
(1.4) The sum of two odd numbers is even
(1.5) a divides b $\Leftrightarrow$ there is an integer x such that $ax=b$

(c) We need to prove $1 \mid a$.

Proof. Since $1 \times a = a$ therefore by definition (1.5) we have $1 \mid a$.

(d) Prove $a \mid a^2$.

Proof. Since $a \times a = a^2$ therefore by definition (1.5) we have $a \mid a^2$.

(e) Prove $a \mid a^n$.

Proof. Since $a \times a^{n-1} = a^n$ which is $a(\text{Integer}) = a^n$ therefore by definition (1.5) we have $a \mid a^n$.

(f) We have to prove $a \mid b$ and $a \mid c \Rightarrow a \mid (b+c)$

Proof. We have $a \mid b$ and $a \mid c$ then by proposition (1.7) we have

$$a \mid (bm + cn)$$

where m and n are arbitrary. Putting $m = n = 1$ we have

$$bm + cn = b(1) + c(1) = b + c$$

the required result, $a \mid (b+c)$.

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(g) Need to prove: $a \mid b$ and $a \mid c \Rightarrow a^2 \mid bc$

Proof. From $a \mid b$ and $a \mid c$ there are integers x and y such that

$$ax = b \text{ and } ay = c$$

Multiplying together gives

$$ax(ay) = bc \quad \text{which simplifies to} \quad a^2(xy) = bc$$

Since $a^2(\text{Integer}) = bc$ therefore $a^2 \mid bc$.

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(h) Need to prove: $ac \mid bc \Rightarrow a \mid b$ where $c \neq 0$

Proof. By using definition (1.5) on $ac \mid bc$ we know there is an integer, x, such that

$$ac(x) = bc$$

Dividing through by c gives

$$a(x) = b \text{ which implies } a \mid b$$

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(i) Prove $a \mid b$ and $\text{c} \mid d \Rightarrow ac \mid bd$

Proof. From $a \mid b$ and $c \mid d$ we have integers x and y such that

$$ax = b \text{ and } cy = d$$

Multiplying together gives

$$ax(cy) = bd$$

$$ac(xy) = bd$$

$$ac(xy) = cd \quad \text{which is } ac(\text{Integer}) = bd$$

(1.5) $a \mid b \Leftrightarrow$ there is an integer x such that $ax = b$

(1.7) If $a \mid b$ and $a \mid c$ then $a \mid (bm + cn)$

By using definition (1.5) in the direction $\Leftarrow$ we have $ac \mid bd$ which is what was required.

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5. (a) We need to prove 'If n is odd then $8 \mid (n^2-1)$'

Proof. We assume n is odd so it can be written as $n = 2m+1$ where m is an integer. Consider n^2-1:

$$n^2-1 = (2m+1)^2 - 1$$

$$= \Big(\underbrace{4m^2+4m+1}_{==(2m+1)^2}\Big) - 1 \qquad \text{[Expanding]}$$

$$= 4m^2+4m = 4m(m+1) \qquad \text{[Factorizing]}$$

We know from the proposition in Question 2(ii) that $m(m+1)$ is even therefore we have

$$n^2-1 = 4\underbrace{m(m+1)}_{\text{Even}}$$

By definition (1.1) we can write $m(m+1) = 2k$ where k is an integer. Hence we have

$$n^2-1 = 4\underbrace{m(m+1)}_{=2k \text{ (Even)}}$$

$$= 4(2k) = 8k$$

Since $n^2-1 = 8k$ which means $8(\text{Integer}) = n^2-1$, therefore $8 \mid (n^2-1)$ and this was what we needed to prove.

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(b) We need to prove 'If n is odd then $32 \mid (n^2+3)(n^2+7)$'

Proof. We assume n is odd so it can be written as $n = 2m+1$ where m is an integer. Consider the first term n^2+3:

$$n^2+3 = (2m+1)^2+3$$

$$= (4m^2+4m+1)+3 \qquad \left[\text{Expanding } (2m+1)^2\right]$$

$$= 4m^2+4m+4 = 4(m^2+m+1) \qquad \text{[Factorizing]}$$

Similarly consider second term n^2+7:

$$n^2+7 = (2m+1)^2+7$$

$$= (4m^2+4m+1)+7$$

$$= 4m^2+4m+8 = 4(m^2+m+2)$$

Multiplying these together gives

$$(n^2+3)(n^2+7) = \underbrace{4(m^2+m+1)}_{=n^2+3}\underbrace{4(m^2+m+2)}_{=n^2+7}$$

$$= \underbrace{16}_{=4\times 4}(m^2+m+1)(m^2+m+2)$$

(1.1) n is an even number $\Leftrightarrow$ $n = 2m$ where m is an integer

Let $m^2+m+1=k$ where k is an integer. Substituting this into the above we have

$$\left(n^2+3\right)\left(n^2+7\right)=16\underbrace{\left(m^2+m+1\right)}_{=k}\left(\underbrace{m^2+m+1}_{=k}+1\right)$$
$$=16k\left(k+1\right)$$

We know from the proposition in Question 2(ii) that $k\left(k+1\right)$ is even therefore we can write $k\left(k+1\right)=2\ell$ where ℓ is an integer. We have

$$\left(n^2+3\right)\left(n^2+7\right)=16\left(2\ell\right) \quad \left[\text{Substituting } k\left(k+1\right)=2\ell\right]$$
$$=32\ell$$

We have $32\left(\text{Integer}\right)=\left(n^2+3\right)\left(n^2+7\right)$. By definition (1.5) we conclude that $32 \mid \left(n^2+3\right)\left(n^2+7\right)$.

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6. Show that if the last digit of an integer n is even then n is even.
Proof. Using the hint we have

$$n=\left(a_m\times10^m\right)+\left(a_{m-1}\times10^{m-1}\right)+\left(a_{m-2}\times10^{m-2}\right)+\ldots+\left(a_2\times10^2\right)+\left(a_1\times10^1\right)+a_0$$
$$=\left[10\left(a_m\times10^{m-1}\right)+10\left(a_{m-1}\times10^{m-2}\right)+10\left(a_{m-2}\times10^{m-3}\right)+\ldots+10\left(a_2\times10^1\right)+10\left(a_1\right)\right]+a_0$$
$$\left[\text{Taking Out a Factor of } 10\right]$$
$$=\left[\begin{array}{l}\left(2\times5\right)\left(a_m\times10^{m-1}\right)+\left(2\times5\right)\left(a_{m-1}\times10^{m-2}\right)+\left(2\times5\right)\left(a_{m-2}\times10^{m-3}\right)+\ldots\\ \qquad+\left(2\times5\right)\left(a_2\times10^1\right)+\left(2\times5\right)\left(a_1\right)\end{array}\right]+a_0$$
$$\left[\text{Rewriting } 10 \text{ as } \left(2\times5\right)\right]$$
$$=2\left[\begin{array}{l}5\left(a_m\times10^{m-1}\right)+5\left(a_{m-1}\times10^{m-2}\right)+5\left(a_{m-2}\times10^{m-3}\right)+\ldots\\ \qquad+5\left(a_2\times10^1\right)+5\left(a_1\right)\end{array}\right]+a_0$$

The last line says $n=2\left[\text{An Integer}\right]+a_0$. We assume a_0 is even because the given proposition says "if the last digit of an integer n is even" and a_0 is the last digit. We can write $a_0=2b$. We have

$$n=2\left[\text{An Integer}\right]+a_0$$
$$=2\left[\text{An Integer}\right]+2b=2\left(\left[\text{An Integer}\right]+1\right)$$

$\left(\text{An Integer}+1\right)=$ (Another Integer) therefore

$$n=2\left(\text{Another Integer}\right)$$

and so by (1.1) we conclude that n is even.

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7. Show that if the last digit of an integer n is odd then n is odd.
Proof. Very similar to the proof of Question 6.

(1.1) n is an even number $\Leftrightarrow$ $n=2m$ where m is an integer
(1.5) $a \mid b$ $\Leftrightarrow$ there is an integer x such that $ax=b$