Complete Solutions to Exercise 1h

5. *Proof*. We first check the proposition for $n=1$

$$a=\frac{a(1-r)}{1-r}=a \qquad \text{[Cancelling } (1-r)\text{'s]}$$

Hence the proposition is true for $n=1$. *What is our next step?*
Assume the proposition is true for $n=k$, that is

$$a+ar+ar^2+\ldots+ar^{k-1}=\frac{a\left(1-r^k\right)}{1-r} \qquad (\$)$$

We need to prove the proposition for $n=k+1$ which is the following;

$$a+ar+ar^2+\ldots+ar^{k-1}+ar^k=\frac{a\left(1-r^{k+1}\right)}{1-r} \qquad (\#)$$

What do we need to prove?
Left Hand Side is equa1 to the Right Hand side of (#). Examining the Left Hand Side of $(\#)$ and using ($) we have

$$a+ar+ar^2+\ldots+ar^{k-1}+ar^k=\underbrace{a+ar+ar^2+\ldots+ar^{k-1}}_{=\frac{a\left(1-r^k\right)}{1-r}\text{ by (\$)}}+ar^k$$

$$=\frac{a\left(1-r^k\right)}{1-r}+ar^k$$

$$=\frac{a\left(1-r^k\right)+ar^k(1-r)}{1-r} \qquad [\text{Common Denominator}]$$

$$=\frac{a-ar^k+ar^k-ar^kr}{1-r} \qquad \left[\begin{array}{l}\text{Expanding Brackets}\\ \text{on Numerator}\end{array}\right]$$

$$=\frac{a-ar^{k+1}}{1-r} \qquad \left[\text{Because } -ar^k+ar^k=0\right]$$

$$=\frac{a\left(1-r^{k+1}\right)}{1-r} \qquad \text{[Factorizing Numerator]}$$

The last line is the Right Hand Side of (#). Therefore we have shown Left Hand Side is equa1 to the Right Hand side of (#). Hence we have our result.

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6. *Proof*. By applying mathematical induction we have:
Check the result is true for $n=1$, that is

$$\sin(x)=\frac{\cos\left(\frac{x}{2}\right)-\cos\left(\frac{2(1)+1}{2}x\right)}{2\sin\left(\frac{x}{2}\right)}$$

$$=\frac{\cos\left(\frac{x}{2}\right)-\cos\left(\frac{3}{2}x\right)}{2\sin\left(\frac{x}{2}\right)} \qquad (\dagger)$$

How do we show the Right Hand Side simplifies to $\sin(x)$ *?*

We need to use the trigonometric identity:

$$\cos(A)-\cos(B)=-2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

on the numerator of (†).

$$\cos\left(\frac{x}{2}\right)-\cos\left(\frac{3x}{2}\right)=-2\sin\left(\frac{x+3x}{4}\right)\sin\left(\frac{x-3x}{4}\right)$$

$$=-2\sin(x)\sin\left(-\frac{x}{2}\right) \qquad [\text{Simplifying}]$$

$$=-2\sin(x)\left(-\sin\left(\frac{x}{2}\right)\right) \qquad [\text{Because } \sin(-\theta)=-\sin(\theta)]$$

$$=2\sin(x)\sin\left(\frac{x}{2}\right)$$

Substituting this into (†) gives

$$\sin(x)=\frac{2\sin(x)\sin\left(\frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)}=\sin(x) \qquad \left[\text{Cancelling } 2\sin\left(\frac{x}{2}\right)\right]$$

Hence the proposition is true for $n=1$. Next we assume the proposition is true for $n=k$:

$$\sin(x)+\sin(2x)+\ldots+\sin(kx)=\frac{\cos\left(\frac{x}{2}\right)-\cos\left(\frac{2k+1}{2}x\right)}{2\sin\left(\frac{x}{2}\right)} \qquad (*)$$

We need to prove the proposition for $n=k+1$, that is

$$\sin(x)+\sin(2x)+\ldots+\sin(kx)+\sin\left((k+1)x\right)=\frac{\cos\left(\frac{x}{2}\right)-\cos\left(\frac{2(k+1)+1}{2}x\right)}{2\sin\left(\frac{x}{2}\right)}$$

$$=\frac{\cos\left(\frac{x}{2}\right)-\cos\left(\left(\frac{2k+3}{2}\right)x\right)}{2\sin\left(\frac{x}{2}\right)} \qquad (**)$$

What do we need to show?

The Left Hand Side is equal to the Right Hand Side of (**). Let's examine the Left Hand Side first.

$$\underbrace{\sin(x)+\sin(2x)+...+\sin(kx)}_{=\frac{\cos\left(\frac{x}{2}\right)-\cos\left(\frac{2k+1}{2}x\right)}{2\sin\left(\frac{x}{2}\right)} \text{ by (*)}}+\sin\left(\left(k+1\right)x\right)=\frac{\cos\left(\frac{x}{2}\right)-\cos\left(\frac{2k+1}{2}x\right)}{2\sin\left(\frac{x}{2}\right)}+\sin\left(\left(k+1\right)x\right)$$

$$=\frac{\cos\left(\frac{x}{2}\right)-\cos\left(\frac{2k+1}{2}x\right)+2\sin\left(\frac{x}{2}\right)\sin\left(\left(k+1\right)x\right)}{2\sin\left(\frac{x}{2}\right)}$$

$$\left[\text{Common Denominator}\right]$$

What do we do next?

We can use the following trigonometric identity on the last term of the numerator:

$$\sin\left(A\right)\sin\left(B\right)=\frac{1}{2}\left[\cos\left(A-B\right)-\cos\left(A+B\right)\right]$$

$$\begin{aligned}2\sin\left(\frac{x}{2}\right)\sin\left(\left(k+1\right)x\right)&=\left[\cos\left(\frac{x}{2}-\left(k+1\right)x\right)-\cos\left(\frac{x}{2}+\left(k+1\right)x\right)\right]\\&=\left[\cos\left(\frac{x}{2}-\frac{\left(2k+2\right)x}{2}\right)-\cos\left(\frac{x}{2}+\frac{\left(2k+2\right)x}{2}\right)\right]\\&=\left[\cos\left(\frac{x-2kx-2x}{2}\right)-\cos\left(\frac{x+2kx+2x}{2}\right)\right]\\&=\left[\cos\left(\frac{-x-2kx}{2}\right)-\cos\left(\frac{3x+2kx}{2}\right)\right]\\&=\left[\cos\left(\frac{x+2kx}{2}\right)-\cos\left(\frac{3x+2kx}{2}\right)\right] \qquad \left[\text{Using } \cos\left(-\theta\right)=\cos\left(\theta\right)\right]\\&=\left[\cos\left(\frac{\left(2k+1\right)x}{2}\right)-\cos\left(\frac{\left(2k+3\right)x}{2}\right)\right]\end{aligned}$$

Substituting this into the above we have

$$\sin(x)+\sin(2x)+...+\sin\left(\left(k+1\right)x\right)=\frac{\cos\left(\frac{x}{2}\right)-\cos\left(\frac{2k+1}{2}x\right)+\left[\cos\left(\frac{\left(2k+1\right)x}{2}\right)-\cos\left(\frac{\left(2k+3\right)x}{2}\right)\right]}{2\sin\left(\frac{x}{2}\right)}$$

$$=\frac{\cos\left(\frac{x}{2}\right)-\cos\left(\frac{\left(2k+3\right)x}{2}\right)}{2\sin\left(\frac{x}{2}\right)}$$

$\left[\text{Because } -\cos\left(\frac{2k+1}{2}x\right)+\cos\left(\frac{(2k+1)x}{2}\right)=0\right]$. Hence we have the Right Hand Side of (**). Therefore we have our required result and the proposition is proved by induction.

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7. *Proof.* We first check the proposition for $n=1$

$$(a+b)^1 = a^1 + b^1 = a+b$$

Hence the proposition is true for $n=1$. *What is our next step?*
Assume the proposition is true for $n=k$, that is

$$(a+b)^k = a^k + ka^{k-1}b + \frac{k(k-1)}{2!}a^{k-2}b^2 + \ldots + b^k \qquad (*)$$

We need to prove the proposition for $n=k+1$ which is the following;

$$(a+b)^{k+1} = a^{k+1} + (k+1)a^{k-1+1}b + \frac{(k+1)((k+1)-1)}{2!}a^{(k+1)-2}b^2 + \ldots + b^{k+1}$$

$$= a^{k+1} + (k+1)a^k b + \frac{(k+1)k}{2!}a^{k-1}b^2 + \ldots + b^{k+1}$$

What do we need to show to prove this?
Left Hand Side is equa1 to the Right Hand Side. *How?*
Using (*) and algebraic manipulation.

$$(a+b)^{k+1} = (a+b)^k(a+b)^1$$

$$= \Bigg(\underbrace{a^k + ka^{k-1}b + \frac{k(k-1)}{2!}a^{k-2}b^2 + \ldots + b^k}_{\text{by } (*)}\Bigg)(a+b)$$

$$= \underbrace{a^k a + ka^{k-1}ba + \frac{k(k-1)}{2!}a^{k-2}b^2 a + \ldots + b^k a}_{\text{Multiplying the Long Bracket by } a} +$$

$$\underbrace{a^k b + ka^{k-1}bb + \frac{k(k-1)}{2!}a^{k-2}b^2 b + \ldots + b^k b}_{\text{Multiplying the Long Bracket by } b}$$

$$= a^{k+1} + ka^k b + \frac{k(k-1)}{2!}a^{k-1}b^2 + \ldots + ab^k +$$

$$a^k b + ka^{k-1}b^2 + \frac{k(k-1)}{2!}a^{k-2}b^3 + \ldots + b^{k+1}$$

[Simplifying by using rules of Indices]

$$= a^{k+1} + (k+1)a^k b + \left[\frac{k(k-1)}{2!} + k\right]a^{k-1}b^2 + \ldots + b^{k+1} \qquad \left[\begin{array}{l}\text{Collecting}\\ \text{like Terms}\end{array}\right]$$

$$= a^{k+1} + (k+1)a^k b + \underbrace{\left[\frac{k(k+1)}{2!}\right]}_{\text{becausc } \frac{k(k-1)}{2!}+k=\frac{k(k+1)}{2!}} a^{k-1}b^2 + \ldots + b^{k+1}$$

Hence we have

$$\left(a+b\right)^{k+1}=a^{k+1}+\left(k+1\right)a^{k}b+\frac{\left(k+1\right)k}{2!}a^{k-1}b^{2}+\ldots+b^{k+1}$$

The required result. We have proven the binomial theorem for all natural numbers.