Complete Solutions to Exercise 1(f)

1. Need to show:

$$\left[\text{not }(P \Rightarrow Q)\right] \equiv \left[P \wedge (\text{not } Q)\right] \qquad [\text{Equivalent}]$$

Column 1	Column 2	Column 3	Column 4	Column 5
P	Q	$P \Rightarrow Q$	not $(P \Rightarrow Q)$	$P \wedge (\text{not } Q)$
T	T	T	F	F
T	F	F	T	T
F	T	T	F	F
F	F	T	F	F

Since the last two columns agree we have the required result.

2. *Proof*. Suppose there is a real number x such that it has two additive inverses call them y and z. Then $y \neq z$ because if $y = z$ then we have a unique additive inverse and there is nothing left to prove. Thus we have

$$x + y = 0 \qquad (\dagger)$$

$$x + z = 0 \qquad (\dagger\dagger)$$

Subtracting the two equations (†) and (††) gives

$$y - z = 0 \quad \Rightarrow \quad y = z$$

Thus we have $y \neq z$ and $y = z$. Contradiction. Therefore every real number has a unique additive inverse.

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3. *Proof*. Suppose $xy = 0$ and both $x \neq 0$ and $y \neq 0$. Multiply both sides of $xy = 0$ by the reciprocal of x. *What is the reciprocal of* x*?*

$$\frac{1}{x}$$

Multiplying $xy = 0$ by $\frac{1}{x}$ gives

$$\frac{1}{x}(xy) = 0$$

$$y = 0 \qquad [\text{Cancelling } x\text{'s}]$$

Contradicting the supposition that $y \neq 0$. Hence the given proposition '$xy = 0$ $\Rightarrow$ $x = 0$ or $y = 0$' is true.

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4. *Proof*. Suppose that n^2 is odd and n is even. We can write n as

$n = 2m$ where m is an integer

Squaring both sides of $n = 2m$ gives

$$n^2 = (2m)^2 = 4m^2 = 2(2m^2)$$

We have $n^2 = 2(\text{Integer})$ which means it is even. Hence we have n^2 is odd and n^2 is even. This contradicts our supposition that 'n^2 is odd and n is even'. Therefore the given proposition 'n^2 is odd $\Rightarrow$ n is odd ' must be true.

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5. *Proof*. Suppose that n^3 is odd and n is even. We can write n as

$$n = 2m \text{ where } m \text{ is an integer}$$

Cubing both sides of $n = 2m$ gives

$$n^3 = (2m)^3 = 8m^3 = 2(4m^3)$$

We have $n^3 = 2(\text{Integer})$ which means it is even. Hence we have n^3 is odd and even which is a contradiction. Our supposition that 'n^3 is odd and n is even' must be false. Therefore the given proposition 'n^3 is odd $\Rightarrow$ n is odd ' must be true.

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6. *Proof*. Suppose that n^3 is even and n is odd. We can write n as

$$n = 2m+1 \text{ where } m \text{ is an integer}$$

Cubing both sides of $n = 2m+1$ gives

$$\begin{aligned} n^3 &= (2m+1)^3 \\ &= 8m^3 + 3(2m)^2 + 3(2m) + 1 \qquad [\text{Expanding by binomial}] \\ &= 2(4m^3) + 12m^2 + 6m + 1 = 2(4m^3 + 6m^2 + 3m) + 1 \end{aligned}$$

We have $n^3 = 2(\text{Integer}) + 1$ which means it is odd. Hence we have n^3 is odd and n^3 is even which is a contradiction. Our supposition that 'n^3 is even and n is odd' must be false. Therefore the given proposition 'n^3 is even $\Rightarrow$ n is even ' must be true.

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7. *Proof*. Suppose that ab is odd and a is even or b is even.
Without loss of generality assume a is even. We can write this as

$$a = 2m \text{ where } m \text{ is an integer}$$

Therefore $ab = 2mb$ which means that ab is even. We have ab is even and ab is odd. Our supposition that 'ab is odd and a is even or b is even' leads to a contradiction therefore the given proposition 'ab is odd $\Rightarrow$ both a is odd and b is odd' is true.

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8. *Proof*. Suppose that ab is even and both a and b are odd.
We can write a and b as

$$a = 2k+1 \text{ and } b = 2m+1 \text{ where } k \text{ and } m \text{ are integers}$$

$$\begin{aligned} ab &= (2k+1)(2m+1) \\ &= 4km + 2k + 2m + 1 = 2(km + k + m) + 1 \end{aligned}$$

$ab = 2(\text{Integer}) + 1$ therefore ab is odd. Since ab is odd and ab is even is a contradiction therefore our supposition 'ab is even and both a and b are odd' is false so the given proposition 'ab is even $\Rightarrow$ a is even or b is even' is true.

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9. *Proof*. Suppose that $\sqrt{6}$ is rational. We can write $\sqrt{6}$ as

$$\frac{p}{q} = \sqrt{6}$$

where p and q have **no** factors in common apart from 1. Multiplying by q and squaring gives

$$\begin{aligned} p &= \sqrt{6}q \\ p^2 &= 6q^2 = 2(3q^2) \qquad [\text{Squaring}] \end{aligned}$$

Since $p^2 = 2(\text{Integer})$ therefore it is even. By lemma (1.12) we have

$$p^2 \text{ is even } \Rightarrow p \text{ is even}$$

We can write $p = 2m$ where *m* is an integer. Substituting this, $p = 2m$, into $p^2 = 6q^2$ gives

$$4m^2 = 6q^2$$

$$2m^2 = 3q^2 \qquad [\text{Dividing by } 2]$$

$3q^2$ is even therefore by the above question 8 we have q^2 is even because 3 is odd. By lemma (1.12)

$$q^2 \text{ is even } \Rightarrow q \text{ is even}$$

Both *p* and *q* are even which means that they have a common factor of 2. We have a contradiction because earlier we said *p* and *q* have **no** factors in common (apart from 1) and now we have shown that they have a common factor of 2. Our initial statement that '$\sqrt{6}$ is rational' must be false therefore $\sqrt{6}$ is irrational.

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10. *Proof*. Suppose $\sqrt[3]{2}$ is rational. We can write this as

$$\frac{p}{q} = \sqrt[3]{2} \text{ where } p \text{ and } q \text{ have } \textbf{no} \text{ factors in common}$$

$$\left(\frac{p}{q}\right)^3 = 2 \Rightarrow p^3 = 2q^3$$

Since $p^3 = 2(\text{Integer})$ therefore p^3 is even. By Question 6 above we have p^3 is even $\Rightarrow$ *p* is even. Writing $p = 2m$ where *m* is an integer gives $p^3 = (2m)^3 = 8m^3$.

Substituting this, $p^3 = 8m^3$, into the above $p^3 = 2q^3$ gives

$$2q^3 = 8m^3$$

$$q^3 = 4m^3 = 2(2m^3) \qquad [\text{Dividing by } 2]$$

Similarly $q^3 = 2(\text{Integer})$ therefore q^3 is even. Again by Question 6 we have q^3 is even $\Rightarrow$ *q* is even.

Both *p* and *q* are even which means that they have a common factor of 2. We have a contradiction because earlier we said *p* and *q* have **no** factors in common (apart from 1) and now we have shown that they have a common factor of 2. Our initial statement that '$\sqrt[3]{2}$ is rational' must be false therefore $\sqrt[3]{2}$ is irrational.

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11. *Proof*. Suppose that $\sqrt{17}$ is rational. We can write $\sqrt{17}$ as

$$\frac{m}{n} = \sqrt{17} \quad \text{where } m \text{ and } n \text{ have } \textbf{no} \text{ factors in common}$$

$$m^2 = 17n^2 \qquad [\text{Squaring}]$$

Therefore $17 \mid m^2$. By hint we have $17 \mid m$ because 17 is prime. We can write *m* as

$$m = 17k \text{ where } k \text{ is an integer}$$

Substituting this, $m = 17k$, into the above $m^2 = 17n^2$ gives

lemma (1.12) n^2 is even $\Rightarrow$ n is even

$$(17k)^2 = 17n^2$$
$$(17)^2 k^2 = 17n^2$$
$$17k^2 = n^2 \qquad [\text{Dividing through by } 17]$$

Hence $17 \mid n^2$ and again by hint $17 \mid n^2 \Rightarrow 17 \mid n$. This means that 17 is a factor of n. Both *m* and *n* have a common factor of 17. We have a contradiction because earlier we said *m* and *n* have **no** factors in common and now we have shown that they have a common factor of 17. Our initial statement that '$\sqrt{17}$ is rational' must be false therefore $\sqrt{17}$ is irrational.

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12. *Proof.* Suppose there are positive integers *a* and *b* such that

$$a^2 - b^2 = 1$$

Since $a^2 - b^2$ is difference of two squares we can write this as

$$a^2 - b^2 = (a-b)(a+b) = 1 \qquad (*)$$

Because *a* and *b* are positive integers therefore $a - b > 0$ which means that $a > b$ or $b < a$. Dividing both sides of (*) by $a+b$ gives

$$a - b = \frac{1}{a+b} < 1 \text{ implies that } a < b+1$$

Combining the two results, $b < a$ and $a < b+1$ we have

$$b < a < b+1$$

which means that a is an integer between b and $b+1$. Since *b* is an integer therefore *a* **cannot be an integer** because it lies between b and $b+1$. Contradicting our supposition that 'there are positive integers such *a* and *b* such that $a^2 - b^2 = 1$'. Hence the given proposition 'that there are **no** positive integer solutions such that

$$a^2 - b^2 = 1\text{'}$$

must be true.

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13. Suppose a is rational and *b* is irrational and $a+b$ is rational. We can write $a+b$ as

$$a + b = \frac{p}{q}$$

$$b = \frac{p}{q} - a = \frac{p - qa}{q}$$

This means that b is rational because we have written it as a fraction of integers. Hence b is rational and irrational. Contradicts our supposition that 'a is rational and *b* is irrational and $a+b$ is rational'. The given proposition 'the sum of a rational and irrational number is irrational' is true.

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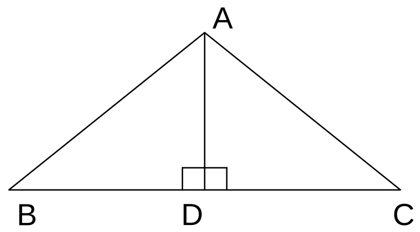
14. Suppose $\angle B = \angle C$ then $AB \neq AC$.



Since $AB \neq AC$ therefore

$$\frac{AD}{AB} \neq \frac{AD}{AC}$$

$$\sin(B) \neq \sin(C) \quad \Rightarrow \quad \angle B \neq \angle C$$

This is a contradiction because $\angle B = \angle C$ and $\angle B \neq \angle C$. Hence the given proposition is true.

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