Tough Nut to Crack - Integration Problem

The integral we are given is:

$$f(x) = \int_{\sqrt{2}}^{2} \frac{1}{x^2\sqrt{(x^2-1)}}\, dx$$

Due to the nature of the integration (two functions being multiplied, without one being a derivative of the other), one method to use is integration by substitution, using trigonometry.

In the book "Engineering Mathematics Through Applications", on page 461 (formula 8.55), it states:

$$when \sqrt{(u^2 - a^2)}\ is\ present, take\ u = a \sec \emptyset$$

As $x^2 - 1$ can be written as $u^2 - 1^2$, we can take $x = \sec \emptyset$. In addition to this, we must also work out $\frac{dx}{d\theta}$. Doing this gives:

$$x = \sec \emptyset$$

$$\frac{dx}{d\theta} = \sec \emptyset \tan \emptyset$$

Therefore, substituting this in to the initial equation gives:

$$f(x) = \int_{\sqrt{2}}^{2} \frac{\sec \emptyset \tan \emptyset}{(\sec \emptyset)^2\sqrt{((\sec \emptyset)^2 - 1)}}\, d\emptyset$$

Cancelling this gives:

$$f(x) = \int_{\sqrt{2}}^{2} \frac{\tan \emptyset}{\sec \emptyset \sqrt{((\sec \emptyset)^2 - 1)}}\, d\emptyset$$

We can then address the trigonometric identity on the denominator. The trigonometric identity $\sec \emptyset = 1 + \tan \emptyset$ can be used to give:

$$f(x) = \int_{\sqrt{2}}^{2} \frac{\tan \emptyset}{\sec \emptyset \sqrt{(1 + \tan^2 \emptyset - 1)}}\, d\emptyset$$

This then all cancels down to give the following:

$$f(x) = \int_{\sqrt{2}}^{2} \frac{\tan \emptyset}{\sec \emptyset \sqrt{(\tan^2 \emptyset)}}\, d\emptyset$$

$$f(x) = \int_{\sqrt{2}}^{2} \frac{\tan \emptyset}{\sec \emptyset \tan \emptyset}\, d\emptyset$$

The $\tan \emptyset$ on both the numerator and denominator cancel out to give:

$$f(x) = \int_{\sqrt{2}}^{2} \frac{1}{\sec \emptyset}\, d\emptyset$$

We can simplify this even further using another trigonometric identity ($\cos \emptyset = \frac{1}{\sec \emptyset}$) to give:

$$f(x) = \int_{\sqrt{2}}^{2} \cos \emptyset \ d\emptyset$$

We can now solve this equation through integration:

$$f(x) = \sin \emptyset$$

However, we must now write the equation in terms of x, as it initially was. First of all, we write out the initial substitution, and then work backwards:

$$x = \sec \emptyset$$

$$\sec^{-1} x = \emptyset$$

This gives:

$$[\sin(\sec^{-1} x)]_{\sqrt{2}}^{2}$$

And so, all that we have to do is to solve the equation. First we can use a circular function formula (this will be shown how it is derived at the end of the solution):

$$\sin(\sec^{-1} x) = \frac{\sqrt{x^2 - 1}}{x}$$

$$= [\frac{\sqrt{x^2 - 1}}{x}]_{\sqrt{2}}^{2}$$

$$= \left(\frac{\sqrt{2^2 - 1}}{2}\right) - \left(\frac{\sqrt{\sqrt{2}^2 - 1}}{\sqrt{2}}\right)$$

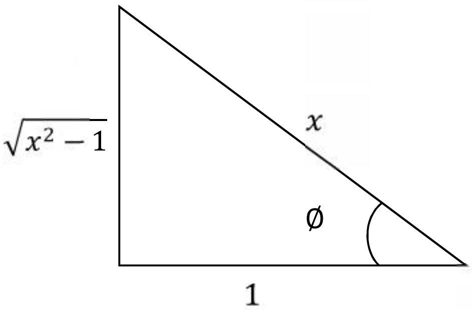
$$= \left(\frac{\sqrt{3}}{2}\right) - \left(\frac{\sqrt{1}}{\sqrt{2}}\right)$$

$$= \frac{\sqrt{3} - \sqrt{2}}{2} = 0.159\ (3d.p.)$$

And so we have our solution.

How to Derive the Circular Function Formula:

Firstly, we can read $\sin(\sec^{-1} x)$ as "the sin of angle $\emptyset$ whose sec is x" ($\sec^{-1} x = \emptyset \rightarrow \sec \emptyset = x$). From this, we can label a right-angle triangle. As $\sec \emptyset = \frac{Hypotenuse}{Adjacent} = \left(\frac{1}{\cos \emptyset}\right)$, we label the triangle as:



As $\frac{Hypotenuse}{Adjacent} = \frac{x}{1} = x$ and $\sqrt{Hypotenuse^2 - Adjacent^2} = Opposite$. Now, we can write $\sin \emptyset$ using this triangle, as $\sin(\sec^{-1} x) = \sin \emptyset$. Therefore:

$$\sin(\sec^{-1} x) = \frac{Opposite}{Hypotenuse} = \frac{\sqrt{x^2 - 1}}{x}$$